Superprotected n-point correlation functions of local operators in $\mathcal{N} = 4$ super Yang-Mills

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Superprotected n-point correlation functions of local operators in $\mathcal{N} = 4$ super Yang-Mills

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Abstract: In this paper we study the n-point correlation functions of two different families of local gauge invariant operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The main idea is to consider the correlation functions of operators which all share a number of supersymmetries irrespective of their relative locations. We achieve this by equipping the operators with explicit space-time dependence. We provide evidence by different methods that these n-point correlators do not receive quantum corrections in perturbation theory and are hence given exactly by their tree-level result. The arguments rely on explicit checks for general four-point correlators, some five-point and six-point correlators and a more abstract calculation based on a novel topological twisting of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, 1/N Expansion, Topological Field Theories

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1 Introduction

Great progress has been achieved in the past few years in precision studies of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory and of the dual string theory on $AdS_5 \times S^5$ [1–3]. The problem of finding the exact anomalous scaling dimensions of local operators has been recast into that of diagonalizing a long-range spin chain model, which — assuming integrability — can be solved for asymptotically long operators by the Bethe ansatz [4–8].[1] The most obvious remaining problem is the understanding of wrapping interactions, which affect short operators at lower loop orders [10, 11].

Beyond that, one would want to go over and obtain all loop results for three-point correlation functions and more generally n-point correlators of local gauge invariant operators. In the case of three-point correlators, they are well understood when all three operators

[1] For reviews see [9].

are chiral primaries (1/2 BPS operators) still from the early days of the AdS/CFT correspondence [12]. These three-point functions are protected from radiative corrections and are given precisely by the free field theory approximation. The case of four-point functions is much more complicated, as they are subject to quantum corrections [13–20], while very little is known about higher-point functions.

The lack of quantum corrections to the two-point and three-point functions of chiral primary operators can be attributed to the fact that all the operators in the correlation function share a number of common supersymmetries. A single operator is annihilated by 24 supercharges: When the operator is located the origin, $x^\mu = 0$, these are all of the superconformal generators (denoted as S) and half of the Poincaré supercharges (denoted as Q). At other space-time positions these are 24 other combinations of these supercharges. The most general combination of three operators of this type at arbitrary space-time positions will still preserve eight supercharges,[2] since each breaks only eight. Four operators, on the other hand, will generically not share any supersymmetries, which is exactly when radiative corrections start to occur.

The object of this paper is to find families of operators which share more supercharges than generic 1/2 BPS operators. One may hope that the correlator of four or more such operators, who share a number of supercharges, will be simpler than that of n-point correllators of generic 1/2 BPS operators. This is indeed true in the two examples of families of operators we present.

A trivial example is the case of operators all preserving the same super-Poincaré generators. If we consider one of the complex scalar fields of the $\mathcal{N} = 4$ supersymmetry multiplet $Z = \Phi^5 + i\Phi^6$ and build operators out of it, then

$$\left\langle \operatorname{Tr} Z^{J_1}(x_1)\, \operatorname{Tr} Z^{J_2}(x_2)\ \cdots\ \operatorname{Tr} Z^{J_n}(x_n)\right\rangle = 0\,. \tag{1.1}$$

This is obvious since they all carry positive charge under a U(1) subgroup of the R-symmetry group. But a similar statement is *almost* true also if there was only $\mathcal{N} = 1$ supersymmetry and no R-charge. In that case chiral primary operators form a ring and do not interact with each other. Their classical n-point function vanishes and they only receive divergent quantum corrections due to instantons. Our examples will share many features with these chiral rings.

In the other examples we present in this paper, the choice of operator is dependent on its spatial position. At different locations the operators will be made of different linear combinations of the scalar fields.

The way we realize this is by taking local operators of the form

$$\operatorname{Tr}\left[u_I(x)\, \Phi^I(x)\right]^J, \tag{1.2}$$

with $u_I(x)$ complex six-vectors. These operators are 1/2 BPS if $u_I(x)\, u_I(x) = 0$ and furthermore, suitable choices of the $u^I(x)$ gives operators that share some conserved supercharges irrespective of the position x^μ in some submanifold of space. In the following

[2] Note that they may break all the Q's and preserve only S's, which is not considered a supersymmetric configuration, but by our counting it would be.

two sections we give two examples of such constructions. The first example in section 2 allows the operators to be at arbitrary points $x^\mu \in \mathbb{R}^4$ and they involve all six real scalars of $\mathcal{N} = 4$ SYM. The second example turns on only three of the scalars and the operators are restricted to $x^\mu \in \mathbb{R}^2$ in space-time.

We study the operators in a variety of ways. After presenting each example we show the supercharges that are preserved by the relevant operators. We then study how the symmetry generators of PSU$(2, 2|4)$ act on the operators. In both cases there are linear combinations of symmetry generators whose action on the operators is particularly simple, these generators arise naturally in topologically twisted versions of $\mathcal{N} = 4$ SYM. In the examples we consider the topological twisting involves conformal generators and not merely the Poincaré group. We will not study the topological twistings in detail, but we expect that a lot of the features that we point out can be proven by use of topological gauge theories.

We then concentrate on perturbative calculations of specific n-point functions of the operators we constructed. Using previously found results for the four-point function of generic chiral primary operators we can immediately show that for our operators there are no perturbative corrections. In a companion paper [21] we develop a simple formula for the one-loop correction to all n-point functions of chiral primary operators. In that paper we use this formula to evaluate some five-point functions and a six-point function at one loop. Here we show that when concentrating on our special operators, these one-loop quantum corrections vanish.

In the next two sections we study the details of the two constructions, relegating more technical details of the supersymmetry algebra to appendices. We conclude in section 4 with a summary of our results and an extensive discussion of possible generalizations and uses of these ideas.

2 Example I: 1/16 BPS n-point functions on $\mathbb{R}^4$

For our first example we take the six real scalars of $\mathcal{N} = 4$ SYM theory $\Phi^1, \ldots, \Phi^6$ and at an arbitrary point $x^\mu \in \mathbb{R}^4$ define the field

$$C(x) = 2ix^\mu \Phi^\mu(x) + i\left(1 - (x^\mu)^2\right)\Phi^5(x) + \left(1 + (x^\mu)^2\right)\Phi^6(x)\,, \tag{2.1}$$

note that this corresponds to the six-vector in (1.2)

$$u_I(x) = \Big(2ix^1\,, 2ix^2\,, 2ix^3\,, 2ix^4\,, i(1 - (x^\mu)^2)\,, 1 + (x^\mu)^2\Big)\,, \tag{2.2}$$

which indeed satisfies $u(x)^2 = 0$. Using $C(x)$ we can then build 1/2 BPS gauge invariant local operators

$$\mathrm{Tr}\; C(x)^J\,. \tag{2.3}$$

In the definition of C we assigned to four of the six scalars a Lorentz index μ, which is the first indication that some topological twisting is involved in the construction. Note that the different terms appearing in the definition have varying scaling dimensions, which could be fixed by adding appropriate powers of an arbitrary length-scale. For simplicity we set this dimensionful constant to unity. The field C was considered in the past in [22, 23],

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for somewhat different motivations. We present our point of view on these operators and will rely on some of the results of [22] below.

When considering the gauge theory on S^4 these operators can also be written in a compact form. Representing the sphere in flat $\mathbb{R}^5$ we have

$$C(x) = i\Phi^m(x)x^m + \Phi^6(x)\,, \qquad m = 1,\cdots,5\,, \qquad (x^m)^2 = 1\,. \tag{2.4}$$

We may also write the sphere as the base of the light-cone in $\mathbb{R}^{5,1}$ and now

$$C(x) \propto x^i\Phi^i(x)\,, \tag{2.5}$$

with $i = 1,\ldots,6$ and in the sixth direction a $(-i)$ is included.

2.1 Supersymmetry

We wish to calculate now the supercharges that are preserved by the field C at an arbitrary point in space. A compact way of writing the general variation of a scalar Φ^i under both the Poincaré and conformal supercharges is as

$$\delta\Phi^i = \bar\psi\rho^i\gamma^5\,\epsilon\,, \qquad \epsilon = \epsilon_0 + \gamma_\mu x^\mu\epsilon_1\,. \tag{2.6}$$

Here ψ is the gluino which transforms in the spinor representation of the Lorentz and SO(6) R-symmetry groups, ρ^i are the SO(6) gamma matrices, while γ_μ are those of the spatial SO(4) and we take them to commute with each-other. ϵ_0 and ϵ_1 are constant 16-component spinors which are the parameters for the super-Poincaré and superconformal transformations respectively. Our notations and details of the superconformal algebra are listed in appendix A.

Applying this to our local field $C(x)$ of (2.1) gives

$$\delta C(x) = \bar\psi\left(2ix^\mu\rho^\mu\gamma^5 + i(1-(x_\mu)^2)\rho^5\gamma^5 + (1+(x_\mu)^2)\rho^6\gamma^5\right)(\epsilon_0 + \gamma_\mu x^\mu\epsilon_1)\,. \tag{2.7}$$

Expanding and separating into terms with different x dependences gives among others, the equations

$$(\rho^6 + i\rho^5)\epsilon_0 = 0\,, \qquad (\rho^6 - i\rho^5)\epsilon_1 = 0\,, \qquad i\rho^\mu\epsilon_0 + \rho^6\gamma^\mu\epsilon_1 = 0\,. \tag{2.8}$$

All the other equations are automatically solved once we impose these conditions, which are also not independent. The first two are a consequence of the last ones, which can be rewritten as

$$\gamma^1\rho^1\epsilon_0 = \gamma^2\rho^2\epsilon_0 = \gamma^3\rho^3\epsilon_0 = \gamma^4\rho^4\epsilon_0 = i\rho^6\epsilon_1\,. \tag{2.9}$$

Since ϵ_0 and ϵ_1 arise from chiral spinors in 10-dimensions,[3] this automatically sets the correct relation between the last two matrices ρ^5 and ρ^6 acting on it, just as the first equation in (2.8). Then ϵ_1 is completely defined in terms on ϵ_0.

The above conditions on ϵ_0 can be rearranged as

$$\gamma^{\mu\nu}\epsilon_0 = -\rho^{\mu\nu}\epsilon_0\,, \qquad \mu,\nu = 1,\cdots,4\,. \tag{2.10}$$

[3] In our conventions $\Gamma^{10}\epsilon_0 = \epsilon_0$ and $\Gamma^{10}\epsilon_1 = -\epsilon_1$ with $\Gamma^{10} = i\gamma^1\gamma^2\gamma^3\gamma^4\rho^1\rho^2\rho^3\rho^4\rho^5\rho^6$.

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Now note that $\gamma^{\mu\nu}$ are the generators of the Lorentz group in the spinor representation while $\rho^{\mu\nu}$ are six out of the 15 generators of the R-symmetry group, also in a spinor representation. This equation suggests taking the diagonal sum of the two groups and imposing that ϵ_0 is a singlet under the diagonal group.

ϵ_0 is the sum of a chiral spinor $\epsilon_{0A}^{+\,\alpha}$ transforming in the $(\mathbf{2},\mathbf{1},\mathbf{4})$ representation of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4)$ and an anti-chiral spinor $\epsilon_0^{-\,\dot\alpha A}$ in the $(\mathbf{1},\mathbf{2},\bar{\mathbf{4}})$ representation. The above equation suggests to break the R-symmetry also to $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$, such that the spinor is decomposed as $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$. We will use dotted lowercase roman indices for $\mathrm{SU}(2)_A$ and undotted ones for $\mathrm{SU}(2)_B$.

Under this decomposition the most general supercharge is generated by

$$\epsilon_{0\,a}^{+\,\alpha} Q_\alpha^a + \dot\epsilon_0^{+\,\alpha\dot a} \dot Q_{\alpha\dot a} - \epsilon_{1\,\dot\alpha a}^{-} \bar S^{\dot\alpha a} - \dot\epsilon_{1\,\dot\alpha}^{-\,\dot a} \dot{\bar S}_{\dot a}^{\dot\alpha} + \epsilon_{1\,\alpha}^{+\,a} S_a^\alpha + \epsilon_0^{-\,\dot\alpha a} \bar Q_{\dot\alpha a} - \dot\epsilon_{1\,\alpha\dot a}^{+} \dot S^{\alpha\dot a} - \dot\epsilon_{0\,\dot a}^{-\,\dot\alpha} \dot{\bar Q}_{\dot\alpha}^{\dot a} \,. \tag{2.11}$$

Details are given in appendix B.

We may now view the above equation (2.10) as relating $\mathrm{SU}(2)_L$ with $\mathrm{SU}(2)_B$ and $\mathrm{SU}(2)_R$ with $\mathrm{SU}(2)_A$, so we need to consider only the spinors with either both dotted or both undotted space-time and R-symmetry indices. Furthermore, the requirement that they are a singlet of the diagonal group means that they can be written as

$$\epsilon_{0a}^{+\,\alpha} = \delta_a^\alpha\, \epsilon_0^+ \,, \qquad \dot\epsilon_{0\dot a}^{-\,\dot\alpha} = \delta_{\dot a}^{\dot\alpha}\, \dot\epsilon_0^- \,, \tag{2.12}$$

where ϵ_0^- and $\dot\epsilon_0^-$ will serve as the two parameters of the unbroken supersymmetries.

ϵ_1 can now be determined through the equation $i\rho^6\epsilon_1 = \gamma^1\rho^1\epsilon_0$. The generator ρ^{16} changes a dotted index into an undotted one, as does the single gamma matrix γ^1. In our notations in appendix A the gamma matrix with lower indices is $\gamma^1_{\dot\alpha\alpha} = i\tau^1$ and in appendix B one finds that $(\rho^{51} + i\rho^{61})^{\,a\dot a} = -i\tau^1$, so

$$\epsilon_{1\alpha}^{+\,a} = \tau^1_{\alpha\dot\alpha}\tau^{1\,a\dot a}\dot\epsilon_{0\dot a}^{-\,\dot\alpha} = \delta_\alpha^a \dot\epsilon_0{}^- \,, \qquad \dot\epsilon_{1\dot\alpha}^{-\,\dot a} = \tau^1_{\alpha\dot\alpha}\tau^{1\,a\dot a}\epsilon_{0a}^{+\,\alpha} = \delta_{\dot\alpha}^{\dot a}\, \epsilon_0^+ \,. \tag{2.13}$$

Plugging this into (2.11) we find that the supercharges that annihilate all of the operators C, regardless of their positions, are

$$\mathcal{Q}^+ = \delta_a^\alpha\, Q_\alpha^a - \delta_{\dot\alpha}^{\dot a}\, \dot{\bar S}_{\dot a}^{\dot\alpha} \,, \qquad \mathcal{Q}^- = \delta_{\dot a}^{\dot\alpha}\, \dot{\bar Q}_{\dot\alpha}^{\dot a} - \delta_\alpha^a\, S_a^\alpha \,. \tag{2.14}$$

While C at a specific position preserves 24 supercharges, like any other chiral field, the fields C all share two supercharges irrespective of their positions. In special cases, when the positions are not totally generic there will be enhanced supersymmetry:

- Clearly at two different points $C(x_1)$ and $C(x_2)$ share sixteen supercharges.

- At three different points operators built out of $C(x_i)$ share only eight supercharges, which is the same as for generic three 1/2 BPS local operators. Furthermore, any three operators define a line or a circle on $\mathbb{R}^4$. If we consider *any* number of operators made of the Cs at arbitrary points along the line/circle they do not break any more of the supersymmetries and still preserve 1/4 of the supercharges.

- Likewise considering C at four points, or at any number of points on an S^2 or an $\mathbb{R}^2$ subspace, will lead to four preserved supercharges.

- Five different operators at generic positions are already the general case and preserve only two supercharges.

2.2 Twisted symmetry

We have seen that operators built out of the field C are all invariant under two supercharges $\mathcal{Q}^\pm$. Here we address how they transform under the remaining symmetry generators.

Some of the symmetry involved in the construction of C is apparent already on a quick inspection of (2.1). We assigned to four of the scalar fields Lorentz indices on $\mathbb{R}^4$, or in the construction based on the light cone (2.5), we assigned a Lorentz index to all six. This suggests that C will transform covariantly when combining R-symmetry rotations and Poincaré and conformal transformations.

Indeed in section 2.1 we saw that the supercharges that annihilate C are singlets of a diagonal subgroup of the $\mathrm{SO}(5,1)$ conformal group and the $\mathrm{SO}(6)$ R-symmetry group.[4] A simple way of finding the twisted symmetry is to take the anti-commutators of $\mathcal{Q}^\pm$ with the other supercharges. As is shown in appendix B, this leads to the combinations of bosonic symmetries (B.13)

$$\begin{aligned}\hat{P}_\mu &= P_\mu + R_{5\mu} + iR_{6\mu}\,,\\ \hat{J}_{\mu\nu} &= J_{\mu\nu} + R_{\mu\nu}\,,\\ \hat{D} &= D + iR_{56}\,,\\ \hat{K}_\mu &= K_\mu + R_{5\mu} - iR_{6\mu}\,.\end{aligned} \tag{2.15}$$

Our construction therefore involves an identification of the R-symmetry group and the space-time group, which is the way one obtains topological theories out of theories with extended supersymmetries. Usually these constructions twist an $\mathrm{SU}(2)$ in space-time by an $\mathrm{SU}(2)$ R-symmetry. Here the twist involves also the conformal generators and as we shall see our other example in section 3 is also associated to topological twistings of a subgroup of the conformal group.

In (A.11), (A.12) the action of the bosonic symmetry generators on scalar fields is written out. From that we can derive the action of the combined generators in (2.15) on our field C, incorporating the explicit space-time dependence

$$\begin{aligned}\hat{P}_\mu\, C &= \partial_\mu C\,,\\ \hat{J}_{\mu\nu}\, C &= (x_\mu\partial_\nu - x_\nu\partial_\mu)C\,,\\ \hat{D}\, C &= x^\mu\partial_\mu C\,,\\ \hat{K}_\mu\, C &= (2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)C\,.\end{aligned} \tag{2.16}$$

[4]These are not the same groups, of course, but both are certain real subgroups of $\mathrm{SL}(4,\mathbb{C})$. We are working mostly at the level of the algebra and are therefore not affected much by this. A more careful treatments is given in [22] where it is argued that the R-symmetry group should really be also $\mathrm{SO}(5,1)$.

Therefore C transforms as a dimension-zero scalar of this twisted conformal group. Indeed its tree-level two-point function is given by

$$\langle C(x_1)\, C(x_2)\rangle_0 = \frac{u_I(x_1)\cdot u_I(x_2)}{(2\pi)^2\,(x_1-x_2)^2} = \frac{1}{2\pi^2}\,, \tag{2.17}$$

suppressing the gauge group indices.

The fact that the symmetry generators arise as anti-commutators with $\mathcal{Q}^\pm$ (B.13) allows us to prove that the n-point function is position independent. Consider the correlator

$$\big\langle \mathrm{Tr}\; C^{J_1}(x_1)\;\mathrm{Tr}\; C^{J_2}(x_2)\;\cdots\;\mathrm{Tr}\; C^{J_n}(x_n)\big\rangle\,. \tag{2.18}$$

We use the fact that $\hat{P}^\mu = \big\{\mathcal{Q}^+,\, Q_\mu\big\}$, that $\mathcal{Q}^+$ annihilates all C's and the Ward-Takahashi identity associated to the symmetry generator $\mathcal{Q}^+$ to derive

$$\begin{aligned}&\frac{\partial}{\partial\, x_1^\mu}\big\langle \mathrm{Tr}\; C^{J_1}(x_1)\;\mathrm{Tr}\; C^{J_2}(x_2)\;\cdots\;\mathrm{Tr}\; C^{J_n}(x_n)\big\rangle\\ &\qquad = \mathcal{Q}^+\big\langle J_1\mathrm{Tr}\;\big[\big\{Q_\mu,\, C\big\}C^{J_1-1}(x_1)\big]\;\;\mathrm{Tr}\; C^{J_2}(x_2)\;\cdots\;\mathrm{Tr}\; C^{J_n}(x_n)\big\rangle = 0\,.\end{aligned} \tag{2.19}$$

This statement is exact regardless of any quantum corrections (including non-perturbative ones).

Furthermore, it was proven in [22] that the action of $\mathcal{N}=4$ SYM theory when restricted to the zero instanton sector is $\mathcal{Q}^\pm$-exact, i.e. $\mathcal{S}_{\text{pert}} = \big\{\mathcal{Q}^\pm,\, \Psi^\pm\big\}$ with some $\Psi^\pm$. This implies that the n-point function receives no perturbative corrections[5]

$$\begin{aligned}&\frac{\partial}{\partial\, g^2_{\text{YM}}}\big\langle \mathrm{Tr}\; C^{J_1}(x_1)\;\mathrm{Tr}\; C^{J_2}(x_2)\;\cdots\;\mathrm{Tr}\; C^{J_n}(x_n)\big\rangle_{\text{pert}}\\ &\qquad\qquad \propto \mathcal{Q}^+\big\langle \Psi^\pm\;\mathrm{Tr}\; C^{J_1}(x_1)\;\mathrm{Tr}\; C^{J_2}(x_2)\;\cdots\;\mathrm{Tr}\; C^{J_n}(x_n)\big\rangle_{\text{pert}} = 0\,.\end{aligned} \tag{2.20}$$

These results are very reminiscent of those for operators in the chiral ring of theories with $\mathcal{N}=1$ supersymmetry. There one can further use cluster decomposition to prove that the n-point function vanishes perturbatively and receives contributions only from the Veneziano-Yankielowicz superpotential.

In our case the theory is conformal, so there is no cluster decomposition. The n-point function is not zero perturbatively, but given by tree-level contractions, as is discussed in the next subsection. We have not evaluated the instanton corrections.

In addition to the two supercharges annihilating the field C, and the fifteen symmetry generators that act on it covariantly (2.15), there are also fifteen more fermionic generators under which it transforms covariantly (B.12). They are given by the sum of the two off-diagonal blocks in (B.2). Together with the bosonic generators (2.15) they form the superalgebra $Q(4)$.

[5] This is true when suitably normalizing the operators to absorb the powers of g_{YM} coming from the free propagators.

2.3 Explicit perturbative calculations

As argued already in the last section, the n-point functions of operators made of powers of C

$$\Big\langle \operatorname{Tr} C^{J_1}(x_1) \operatorname{Tr} C^{J_2}(x_2) \ldots \operatorname{Tr} C^{J_n}(x_n) \Big\rangle, \tag{2.21}$$

receive no radiative corrections in perturbation theory and may thus be called "superprotected". This is a property known to be true for two-pint and three-point functions of *all* chiral primary operators, the novelty here is that it extends to n-point functions of the special chiral primary operators made of the field C.

The argument given in the preceding section for the vanishing of all perturbative corrections to the n-point function is based on the proof of [22] that the action is $Q^{\pm}$ exact. We want to back up this elegant formal argument through explicit computations of the first quantum correction to all n-point functions and all the perturbative corrections to the four-point function of these operators (all with the same J). These considerations will also be of later use in section 3.

In [21] we derive a compact expression for the planar one-loop quantum correction to all n-point functions of operators of the form

$$\mathcal{O}^u_J(x) = \operatorname{Tr}\left[u_I\, \Phi^I(x)\right]^J, \tag{2.22}$$

where the u_I are arbitrary complex six-component vectors obeying $u_I\, u_I = 0$. This makes $\mathcal{O}^u_J$ a chiral primary.

The one-loop correction to the n-point function is written as a sum over all possible choices of four of the operators, with labels i, j, k and l. One field from each of these operators interacts through a combined four-point vertex D_{ijkl} and the rest of the fields of these four operators have to be contracted with all the other operators in a planar way (on a disc, with these four operators on the boundary). This can be written as

$$\Big\langle \mathcal{O}^{u_1}_{J_1} \cdots \mathcal{O}^{u_n}_{J_n} \Big\rangle_{\text{1-loop}} = \sum_{i,j,k,l} J_i J_j J_k J_l\, D_{ijkl} \Big\langle \mathcal{O}^{u_i}_{J_i-1} \mathcal{O}^{u_j}_{J_j-1} \mathcal{O}^{u_k}_{J_k-1} \mathcal{O}^{u_l}_{J_l-1} \Big| \prod_{p\neq i,j,k,l} \mathcal{O}^{u_p}_{J_p} \Big\rangle_{\text{tree, disc}} \tag{2.23}$$

The effective interaction vertex D is[6]

$$D_{1234} = \frac{\lambda}{32\pi^2}\, \Phi(s,t) \big(2\,[13][24] + (s-1-t)[14][23] + (t-1-s)\,[12][34]\big), \tag{2.24}$$

where $[ij]$ are the tree level contractions (without gauge-group indices), while s and t are the cross-ratios

$$[ij] \equiv \frac{1}{(2\pi)^2}\, \frac{u^i_I \cdot u^j_I}{x^2_{ij}}\,, \qquad s = \frac{x^2_{12}\, x^2_{34}}{x^2_{13}\, x^2_{24}}\,, \qquad t = \frac{x^2_{14}\, x^2_{23}}{x^2_{13}\, x^2_{24}}\,, \qquad x_{ij} \equiv x_i - x_j\,, \tag{2.25}$$

and $\Phi(s,t)$ is the scalar box integral [24]

$$\Phi(s,t) = \frac{x^2_{13}\, x^2_{24}}{\pi^2} \int d^4x_5\, \frac{1}{x^2_{15}\, x^2_{25}\, x^2_{35}\, x^2_{45}}\,. \tag{2.26}$$

[6] For clarity we sometimes replace the general indices $ijkl$ with 1234.

One last thing to note, in equation (2.23) one should sum over three inequivalent orders of the operators: $ijkl$, $ikjl$ and $iklj$, since the tree level disc amplitudes with these orderings are generically different. It makes some sense to combine all these terms together, since $D_{1234} + D_{1324} + D_{1243} = 0$, which allows to simplify some expressions, but this is not necessary for the current calculation. It will be important in section 3.

With this result it is easy to prove that there are no one-loop corrections to the n-point function of operators made of the field C. In this case we have (2.2) that $u_I^i = u_I(x_i)$ depends on the position x_i. This gives the inner product $u_I^i \cdot u_I^j = 2x_{ij}^2$ and hence the free-field contractions are all constant

$$[ij] = \frac{1}{2\pi^2}\,. \tag{2.27}$$

Plugging into (2.24) we find that $D_{ijkl} = 0$, so there are no one-loop corrections to *any* of the n-point functions of our operators.

In the case of four-point functions, we can extend this to an *all*-loop statement, relying on the results of Arutyunov, Dolan, Osborn and Sokatchev [18, 19].

Based on superconformal symmetry and additional dynamical input these authors showed that the all-loop quantum corrections to the four-point amplitude of general chiral primaries of weight J are of a factorized, universal form

$$\left\langle \mathcal{O}_J^{u_1}(x_1)\,\mathcal{O}_J^{u_2}(x_2)\,\mathcal{O}_J^{u_3}(x_3)\,\mathcal{O}_J^{u_4}(x_4)\right\rangle_{\text{quant}} = \mathcal{R}(s,t;\mathcal{X},\mathcal{Y},\mathcal{Z})\,\mathcal{F}_J(s,t;\mathcal{X},\mathcal{Y},\mathcal{Z};\lambda)\,, \tag{2.28}$$

where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are the pair-wise contractions

$$\mathcal{X} = [12][34]\,, \qquad \mathcal{Y} = [13][24]\,, \qquad \mathcal{Z} = [14][23]\,. \tag{2.29}$$

The important ingredient in (2.28) is $\mathcal{R}$, the *universal polynomial prefactor* which is independent of J or λ. It is given by the simple combination

$$\begin{aligned} \mathcal{R} &= s\,(\mathcal{Y}-\mathcal{X})(\mathcal{Z}-\mathcal{X}) + t\,(\mathcal{Z}-\mathcal{X})(\mathcal{Z}-\mathcal{Y}) + (\mathcal{Y}-\mathcal{X})(\mathcal{Y}-\mathcal{Z}) \\ &= \frac{16}{\lambda\,\Phi(s,t)}\,(\mathcal{Y}D_{1234} + \mathcal{X}D_{1324} + \mathcal{Z}D_{1243})\,. \end{aligned} \tag{2.30}$$

Moreover the functions $\mathcal{F}_J$ are known up to two-loop order for $J \leq 4$ [20].

Clearly in our case

$$\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \frac{1}{4\pi^4}\,, \tag{2.31}$$

so $\mathcal{R} = 0$ and therefore there are no radiative corrections to the four-point functions.

3 Example II: 1/8 BPS n-point functions on $\mathbb{R}^2$

We now turn to the discussion of our second example for a superprotected operator. If we restrict the operator C from section 2 to the (x^1, x^2) plane (i.e. $x^3 = x^4 = 0$) it is

$$C = 2ix^1\Phi^1 + 2ix^2\Phi^2 + i(1-(x^\mu)^2)\Phi^5 + (1+(x^\mu)^2)\Phi^6\,. \tag{3.1}$$

These operators will share four supercharges, twice as many as the most general operators on $\mathbb{R}^4$. We present in this section another construction of local operators on this plane made out of only three of the scalars Φ^1, Φ^2 and Φ^3, which will also share four supercharges.

Using the complex coordinates $w = x^1 + ix^2$ and $\bar{w} = x^1 - ix^2$ define

$$Z = i(1-\bar{w}^2)\Phi^1 + (1+\bar{w}^2)\Phi^2 - 2i\bar{w}\Phi^3\,, \tag{3.2}$$

which corresponds to the choice in (1.2)

$$u_I(\bar{w}) = \Big(i(1-\bar{w}^2)\,, 1+\bar{w}^2\,, -2i\bar{w}\,, 0\,, 0\,, 0\Big). \tag{3.3}$$

As before we use Z to construct gauge invariant local operators

$$\mathrm{Tr}\ Z^J(w,\bar{w})\,, \tag{3.4}$$

at arbitrary positions on $\mathbb{R}^2$.

While the definition of Z is different from the restriction of C to generic points on $\mathbb{R}^2$, if we restrict both to a line, they are the same up to the choice of scalar fields. For real w, for example, Z in (3.1) is the same as C (2.1) with $(\Phi^1,\,\Phi^2,\,\Phi^3) \to (\Phi^5,\,\Phi^6,\,-\Phi^1)$. Indeed while generically all Cs share four supercharges, along a line or a circle they share eight.

A nice realization of the same operators Z shows up when considering three scalar fields on S^2. Using the indices $i,j,k = 1,2,3$ both for unit three-vectors and for the three scalars, we may define the following scalar field

$$Z^i = (\delta^{ij} - x^i x^j)\Phi^j + i\varepsilon_{ijk}x^j\Phi^k\,. \tag{3.5}$$

We study operators built out of this field in appendix D, where we explain the spurious superscript in Z^i and how it is related to Z in (3.2).

3.1 Supersymmetry

Examining the invariance of these operators under supersymmetry leads to the equations

$$\left(\rho^- - \bar{w}\rho^3 - \bar{w}^2\rho^+\right)\left(\epsilon_0 + (w\gamma^- + \bar{w}\gamma^+)\epsilon_1\right) = 0\,, \tag{3.6}$$

where we defined $\rho^\pm = (\rho^1 \pm i\rho^2)/2$ and $\gamma^\pm = (\gamma^1 \pm i\gamma^2)/2$.

Requiring that this is satisfied for all w and $\bar{w}$ leads to the independent equations

$$\rho^3\epsilon_0 - \rho^-\gamma^+\epsilon_1 = 0\,, \qquad \rho^+\epsilon_0 + \rho^3\gamma^+\epsilon_1 = 0\,, \qquad \gamma^-\epsilon_1 = 0\,. \tag{3.7}$$

We can isolate the following conditions on ϵ_1

$$\gamma^-\epsilon_1 = \rho^3\rho^+\epsilon_1 = 0\,. \tag{3.8}$$

As in section 2.1, it proves useful again to consider the breaking of the R-symmetry group $\mathrm{SO}(6) \to \mathrm{SU}(2)_{A'} \times \mathrm{SU}(2)_{B'}$, but in a different way than discussed there. For the case at hand we take $\mathrm{SU}(2)_{A'}$ to rotate the first three scalars Φ^1, Φ^2 and Φ^3. $\mathrm{SU}(2)_{B'}$ will rotate the remaining three scalars, which do not appear in Z and therefore we will not find any constraints associate to it. Under this breaking, which is discussed in detail in appendix C, the **4** of SO(6) is decomposed into the $(\mathbf{2},\mathbf{2})$ of the broken group.[7] The index A of SU(4) is replaced by the pair $\dot a a$, with the dotted and undotted indices representing $\mathrm{SU}(2)_{A'}$ and $\mathrm{SU}(2)_{B'}$ respectively. The anti-symmetric ρ^{ij} with $i,j=1,2,3$ are the generators of $\mathrm{SU}(2)_{A'}$ and can be written in terms of Pauli matrices. In addition we consider the chiral decomposition of the spinors under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ with indices α and $\dot\alpha$ respectively.

Under this decomposition the chiral and anti-chiral parts of ϵ_0 have the indices $\epsilon_{0\,\dot a a}^{+\,\alpha}$ and $\epsilon_0^{-\,\dot\alpha\dot a a}$ and of ϵ_1 they are $\epsilon_{1\,\alpha}^{+\,\dot a a}$ and $\epsilon^-_{1\,\dot\alpha\dot a a}$. The most general supersymmetry transformation is then generated by

$$\epsilon_{0\,\dot a a}^{+\,\alpha} Q_\alpha^{\ \dot a a} + \epsilon_0^{-\,\dot\alpha\dot a a}\bar Q_{\dot\alpha\dot a a} + \epsilon_{1\,\alpha}^{+\,\dot a a} S^\alpha_{\ \dot a a} - \epsilon^-_{1\,\dot\alpha\dot a a}\bar S^{\dot\alpha\dot a a}\,. \tag{3.9}$$

The specific choice of gamma matrices in (A.5) is such that

$$\gamma^+_{\alpha\dot\alpha} = i\delta^1_\alpha\delta^{\dot 2}_{\dot\alpha}\,,\qquad \gamma^-_{\alpha\dot\alpha} = i\delta^2_\alpha\delta^{\dot 1}_{\dot\alpha}\,,\qquad \gamma^{+\dot\alpha\alpha} = -i\delta^{\dot\alpha}_{\dot 1}\delta^\alpha_2\,,\qquad \gamma^{-\dot\alpha\alpha} = -i\delta^{\dot\alpha}_{\dot 2}\delta^\alpha_1\,. \tag{3.10}$$

Likewise (C.4)

$$(\rho^{3+})^{\dot a}{}_{\dot b} = \delta^{\dot a}_{\dot 1}\delta^{\dot 2}_{\dot b}\,,\qquad (\rho^{3-})^{\dot a}{}_{\dot b} = -\delta^{\dot a}_{\dot 2}\delta^{\dot 1}_{\dot b}\,. \tag{3.11}$$

The equation $\gamma^-\epsilon_1 = 0$ means that for the chiral component, $\epsilon_{1\,\alpha}^{+\,\dot a a}$, the subscript α has to be 2 and for the anti-chiral part $\dot\alpha = \dot 1$. The equation $\rho^{3+}\epsilon_1 = 0$ means that the superscript $\dot a = \dot 1$, and as a subscript $\dot a = \dot 2$. Therefore

$$\epsilon^{+\,\dot a a}_{1\,\alpha} = \delta^2_\alpha\delta^{\dot a}_{\dot 1}\epsilon^{+\,a}\,,\qquad \epsilon^-_{1\,\dot\alpha\dot a a} = \delta^{\dot 1}_{\dot\alpha}\delta^{\dot 2}_{\dot a}\epsilon^-_a\,, \tag{3.12}$$

with arbitrary $\epsilon^{+\,a}$ and ϵ^-_a. Now we can use the first equation in (3.7) to solve for ϵ_0

$$\epsilon_0^{-\,\dot\alpha\dot a a} = i\delta^{\dot\alpha}_{\dot 1}\delta^{\dot a}_{\dot 2}\epsilon^{+\,a}\,,\qquad \epsilon^{+\,\alpha}_{0\,\dot a a} = i\delta^\alpha_2\delta^{\dot 1}_{\dot a}\epsilon^-_a\,, \tag{3.13}$$

Using (3.9), this gives the four independent supersymmetry generators

$$\mathcal{Q}^+_a = \bar Q_{\dot 1\dot 2 a} - iS^2{}_{\dot 1 a}\,,\qquad \mathcal{Q}^{-\,a} = Q_2{}^{\dot 1 a} + i\bar S^{\dot 1\dot 2 a}\,. \tag{3.14}$$

Note that the supercharges mix S and Q generators of different chirality.

The supercharges should commute to symmetries of the operators, which are the rotation in the transverse plane and $\mathrm{SU}(2)_{B'}$ rotations

$$\left\{\mathcal{Q}^+_a\,,\mathcal{Q}^{-\,b}\right\} = -i\delta^b_a(J^2{}_2 - \bar J^{\dot 1}{}_{\dot 1}) - iT^b{}_a. \tag{3.15}$$

Indeed the trace part is the rotation in the (x_3, x_4) plane and the triplet of a and b are the $\mathrm{SU}(2)_{B'}$ generators.

[7] The breaking in section 2.1 is such that $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$.

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3.2 Perturbative calculation

We want to calculate n-point correlators of operators built out of the field Z

$$\Big\langle \mathrm{Tr}\, Z^{J_1}(w_1,\bar w_1)\ \mathrm{Tr}\, Z^{J_2}(w_2,\bar w_2)\ \cdots\ \mathrm{Tr}\, Z^{J_n}(w_n,\bar w_n)\Big\rangle\,, \tag{3.16}$$

with all n-points $(w_i,\bar w_i)$ lying in the plane.

At tree level we should consider all possible contractions of Z fields. Now using (3.3) we have $u_I(\bar w_i)\cdot u_I(\bar w_j)=2(\bar w_i-\bar w_j)^2$. The free-field contractions are therefore given by

$$[12]\equiv\langle Z(w_1,\bar w_1)\,Z(w_2,\bar w_2)\rangle=\frac{1}{2\pi^2}\frac{\bar w_{12}}{w_{12}}\,,\qquad w_{ij}\equiv w_i-w_j\,, \tag{3.17}$$

where as before we suppressed gauge indices.

This two-point function is equivalent to that of a (matrix valued) field in a two-dimensional conformal field theory with conformal weights $(\frac{1}{2},-\frac{1}{2})$. We discuss the transformation properties of the field Z under twisted conformal symmetries in the next subsection.

The operators $\mathrm{Tr}\, Z^J$ are chiral primary operators of $\mathcal{N}=4$ SYM, so the two and three-point functions do not receive quantum corrections and are given by considering all possible free-field contractions (3.17).

Unlike the case of the operators in section 2, for the operators made of the field Z on $\mathbb{R}^2$, we do not have a general proof for the vanishing of the quantum corrections. It may be possible to show that the action is exact under the supersymmetries that annihilate Z, which would prove this statement.

Instead we proceed here to study the correlation functions of these operators in special cases. First we consider the four-point functions, based on the general results of [18, 19]. Then we turn to some specific examples of five and six-point functions of operators of low dimension and show by explicit calculations performed in our companion paper [21] that the *one*-loop correction vanishes.

The first interesting quantity is the four-point function of these operators. As discussed in section 2.3, the key ingredients that appear in this calculation are the pairwise contractions (2.29)

$$\mathcal{X}=[12][34]\,,\qquad \mathcal{Y}=[13][24]\,,\qquad \mathcal{Z}=[14][23]\,. \tag{3.18}$$

The two other ingredients are the conformal invariant cross ratios (2.25), which may also be expressed in terms of a complex number μ

$$s=\frac{x_{12}^2\,x_{34}^2}{x_{13}^2\,x_{24}^2}=\mu\bar\mu\,,\qquad t=\frac{x_{14}^2\,x_{23}^2}{x_{13}^2\,x_{24}^2}=(1-\mu)(1-\bar\mu)\,. \tag{3.19}$$

Using these, the universal polynomial prefactor (2.30) of [18, 19] takes the factorized form

$$\begin{aligned}\mathcal{R}&=s\,(\mathcal{Y}-\mathcal{X})\,(\mathcal{Z}-\mathcal{X})+t\,(\mathcal{Z}-\mathcal{X})\,(\mathcal{Z}-\mathcal{Y})+(\mathcal{Y}-\mathcal{X})\,(\mathcal{Y}-\mathcal{Z})\\&=\Big(\mu\,(\mathcal{X}-\mathcal{Z})+\mathcal{Z}-\mathcal{Y}\Big)\Big(\bar\mu\,(\mathcal{X}-\mathcal{Z})+\mathcal{Z}-\mathcal{Y}\Big)\,.\end{aligned} \tag{3.20}$$

So far this expression does not assume our specific operators, it only uses the complex representation of the cross-ratios (3.19).

In our case μ can be written explicitly as the cross ratio of the four points w_i on the complex plane[8]

$$\mu = \frac{w_{12}\, w_{34}}{w_{13}\, w_{24}} \tag{3.21}$$

We note now that for our special operators Z, the pair-wise contractions $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are related to the cross-ratios by

$$\frac{\mathcal{X}}{\mathcal{Y}} = \frac{\bar{\mu}}{\mu}\,, \qquad \frac{\mathcal{Z}}{\mathcal{Y}} = \frac{1-\bar{\mu}}{1-\mu}\,. \tag{3.22}$$

With this we find the 'magical' identity

$$\mu\,(\mathcal{X} - \mathcal{Z}) + \mathcal{Z} - \mathcal{Y} = 0\,, \tag{3.23}$$

so

$$\mathcal{R} = 0\,, \tag{3.24}$$

and therefore all the four-point functions do not receive any quantum corrections in perturbation theory!

For correlation functions beyond the four-point function we do not have general results. We did calculate, though, several five and six-point functions at one-loop order and found that the quantum corrections vanish, suggesting that this might be a general property of all n-point functions.

The calculation is done by using the results of [21], where the one loop correction to the n-point function of chiral primary operators is written as a sum of insertions of an effective four-scalar vertex D_{ijkl} into tree-level disc amplitudes (2.24).

Using the complex cross-ratio μ, the function D_{1234} can be written as

$$D_{1234} = \frac{\lambda}{32\pi^2}\,\Phi(s,t)\big(2\,[13][24] - (2-\mu-\bar{\mu})[14][23] - (\mu+\bar{\mu})[12][34]\big)\,. \tag{3.25}$$

In our case we can furthermore use (3.22) to simplify this to

$$D_{1234} = -\frac{\lambda}{32\pi^2}\,\Phi(s,t)\,\mathcal{Y}\,\frac{(\mu-\bar{\mu})^2}{\mu(1-\mu)}\,. \tag{3.26}$$

$\Phi(s,t)$ is a transcendental function of the cross-ratios (2.26), and therefore the sum over different D_{ijkl} insertions in (2.23) is over different transcendental functions among which there cannot be cancelations. The exception are terms with the same four vertices but with a different ordering. It is always true that $D_{1234} + D_{1324} + D_{1243} = 0$, but using the expression in (3.26) valid for our operators we find furthermore that D satisfies the modular relations

$$D_{1234} = -\frac{1}{\mu}\,D_{1324} = -\frac{1}{1-\mu}\,D_{1243}\,, \tag{3.27}$$

[8]In general, any four points sit on a sphere or a plane in $\mathbb{R}^4$ and defining μ by solving (3.19) will give the complex conformal cross-ratio of these points with the natural complex-structure on that sphere/plane.

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Let us now examine the particular example of the minimal five-point function, that of operators of dimension two, the general insertion formula (2.23) gives [21]

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} = -\,32\,\Big(&D_{1234}\big([13][52][45]+[15][53][24]\big)\\ &+D_{1324}\big([12][35][54]+[15][52][34]\big)\\ &+D_{1243}\big([14][25][53]+[15][54][23]\big)\\ &+\text{cyclic permutations of }(12345)\Big).\end{aligned} \tag{3.28}$$

Using the modular property (3.27) allows us to simplify the three terms we have written explicitly in (3.28), which add up to

$$\begin{aligned}D_{1234}\Big([13][52][45]+[15][53][24]-\mu\big([12][35][54]+[15][52][34]\big)\\ -(1-\mu)\big([14][25][53]+[15][54][23]\big)\Big)\end{aligned} \tag{3.29}$$

By an explicit calculation, plugging in the value of the tree-level contractions (3.17), we find that this sum vanishes. Hence there are no one-loop correction to this five-point function.

Furthermore, in [21] several other examples of five-point functions of operators of total dimension up to sixteen were calculated and it was shown that they can always be written as a sum of six terms. One is proportional to (3.28) and the rest are proportional to $\mathcal{R}$ (3.20) (with the five different choices of four points). Since these constituents vanish for operators made solely of Z, the one-loop corrections to all these five-point functions vanish. If such a decomposition of the five-point amplitude generalizes also for chiral primary operators of higher dimension, it would then immediately imply the vanishing of the one-loop correction to any five-point function made of Z.

We also computed in [21] one six-point function, that of six chiral primary operators of dimension two. It is written again as a sum similar to (3.28) and by plugging in our choice of operators, using the modular relation (3.27) we get a sum of fifteen terms similar to (3.29) (but with eighteen terms instead of six and each made of four tree-level contractions, instead of three). By direct calculation we found that this vanishes. It would be interesting to understand higher-point functions, both as to their factorization into n-point function of operators of dimension two, and to the vanishing of the analogs of (3.29). We leave this for future explorations.

We would like to stress again that unlike the field C of section 2, the correlators of operators made of Z are not constant, rather they involve the ratio of the anti-holomorphic and holomorphic distances between the points.

3.3 Twisted symmetry

It is clear that in addition to the four supersymmetries calculated in section 3.1, the field Z is invariant under J_{34}, the rotation that leaves the plane invariant, as well as under the action of the three generators of the R-symmetry group that act on the three remaining scalars Φ^4, Φ^5 and Φ^6, which we dubbed $\mathrm{SU}(2)_{B'}$.

Beyond that, being restricted to the plane, Z transforms in representations of $\mathrm{SL}(2,\mathbb{C})$, of rigid conformal transformations on the plane generated by P_1, P_2, K_1, K_2, J_{12} and D.

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Likewise, since it involve Φ^1, Φ^2 and Φ^3, Z can be classified in terms of the $\mathrm{SU}(2)_{A'}$ group that rotates them, generated by R_{12}, R_{23} and R_{31}.

Consider the three generators of the holomorphic $\mathrm{SL}(2,\mathbb{R})$

$$L_1 = \frac{1}{2}(P_1 - iP_2)\,, \qquad L_0 = \frac{1}{2}(D - iJ_{12})\,, \qquad L_{-1} = \frac{1}{2}(K_1 + iK_2)\,. \tag{3.30}$$

These operators act on Z by

$$L_1\, Z = \partial_w Z\,, \qquad L_0\, Z = w\,\partial_w Z + \frac{1}{2} Z\,, \qquad L_{-1}\, Z = w^2\,\partial_w Z + wZ\,. \tag{3.31}$$

Z therefore transforms as a weight $1/2$ primary field of this group.

Since some of the other symmetry generators do not close on Z, it will prove useful to define two more fields made of the same three scalars

$$Y = -i\bar{w}\Phi^1 + \bar{w}\Phi^2 - i\Phi^3\,, \qquad W = -i\Phi^1 + \Phi^2\,. \tag{3.32}$$

The transformation rules of Y and W under L_i are identical to that of Z. This is clearly the same behavior as for any of the scalar fields, since there is no explicit w dependence in the definitions of Z, Y and W

The rest of the symmetry generators can be organized as

$$R_+ = -i(R_{23} + iR_{31})\,, \qquad R_0 = iR_{12}\,, \qquad R_- = i(R_{23} - iR_{31})\,, \tag{3.33}$$

$$\bar{L}_1 = \frac{1}{2}(P_1 + iP_2)\,, \qquad \bar{L}_0 = \frac{1}{2}(D + iJ_{12})\,, \qquad \bar{L}_{-1} = \frac{1}{2}(K_1 - iK_2)\,, \tag{3.34}$$

Their action on the fields Z, Y and W are not too simple and are given in the appendix, see (C.11) (C.10).

A natural thing to try is to take the linear combination of $\bar{L}$ and R. Consider for example

$$\dot{L}_1 = \bar{L}_1 + R_+\,, \qquad \dot{L}_0 = \bar{L}_0 + R_0\,, \qquad \dot{L}_{-1} = \bar{L}_{-1} + R_-\,. \tag{3.35}$$

Their action on Z is given by

$$\dot{L}_1\, Z = \partial_{\bar{w}} Z\,, \qquad \dot{L}_0\, Z = \bar{w}\,\partial_{\bar{w}} Z - \frac{1}{2} Z\,, \qquad \dot{L}_{-1}\, Z = \bar{w}^2\,\partial_{\bar{w}} Z - \bar{w} Z\,. \tag{3.36}$$

Z therefore transforms as a weight $-1/2$ field of this twisted anti-holomorphic $\mathrm{SL}(2,\mathbb{R})$. The action on Y and W is given in (C.13). Y has weight $1/2$ and W has weight $3/2$, but they are not primaries, since there are additional terms in the action of $\dot{L}_{-1}$.

It will turn out that a different combination of the anti-holomorphic symmetry generators and rotations is related to supersymmetries preserved by the operators Z. These are

$$\hat{L}_1 = \bar{L}_1 + \frac{1}{2}R_+\,, \qquad \hat{L}_0 = \bar{L}_0 + \frac{1}{2}R_0\,, \qquad \hat{L}_{-1} = \bar{L}_{-1} + \frac{1}{2}R_-\,. \tag{3.37}$$

Note that because of the factor of $1/2$ those generators do not close onto themselves, and do not form an $\mathrm{SL}(2,\mathbb{R})$ algebra.

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The action of these operators on Z is

$$\hat{L}_1 Z = \partial_{\bar{w}} Z - Y\,, \qquad \hat{L}_0 Z = \bar{w}\,\partial_{\bar{w}} Z - \bar{w} Y\,, \qquad \hat{L}_{-1} Z = \bar{w}^2\,\partial_{\bar{w}} Z - \bar{w}^2 Y\,. \tag{3.38}$$

Under this twisting Z has dimension zero, but has these extra terms proportional to Y in the action of $\hat{L}$. The actions on Y and W are given in (C.15), where Y has dimension 1/2 and W dimension one.

To see how these symmetry generators come about, consider the anti-commutators of $\mathcal{Q}^{\pm}$ with all the other supercharges which will generate some of the bosonic symmetries of the theory. Most of these symmetries will map our operators to others, taking them away from the (x_1, x_2) plane or turning on the three remaining scalars. But the following combinations map our operators to themselves

$$\begin{aligned}
\big\{\mathcal{Q}_a^+\,, iQ_2{}^{\mathrm{i}a} + \bar{S}^{\mathrm{i}\dot{2}a}\big\} &= 2\big(J^2{}_2 + \bar{J}^{\mathrm{i}}{}_{\mathrm{i}} + D\big) + \dot{T}^{\mathrm{i}}{}_{\mathrm{i}} - \dot{T}^{\dot{2}}{}_{\dot{2}} = 2(D + iJ_{12} + iR_{12}) = 4\hat{L}_0\,,\\
\big\{\mathcal{Q}_a^+\,, -iQ_2{}^{\dot{2}a}\big\} &= -2iP_{2\mathrm{i}} - \dot{T}^{\dot{2}}{}_{\mathrm{i}} = P_1 + iP_2 - i(R_{23} + iR_{31}) = 2\hat{L}_1\,,\\
\big\{\mathcal{Q}_a^+\,, -\bar{S}^{\mathrm{i}\mathrm{i}a}\big\} &= 2iK^{\mathrm{i}2} + \dot{T}^{\mathrm{i}}{}_{\dot{2}} = K_1 - iK_2 + i(R_{23} - iR_{31}) = 2\hat{L}_{-1}\,.
\end{aligned} \tag{3.39}$$

Similar expressions exist for $\mathcal{Q}^{-a}$ giving the same combinations of symmetry generators on the right-hand side. Note that these symmetries include both space-time generators and R-rotations of $\mathrm{SU}(2)_{A'}$ and are the second twisting discussed above. Their action on the fields Z, Y and W are given in (3.38) and (C.15).

These twisted symmetry generators can be used to find extra relations among the n-point function of operators with $\mathrm{Tr}\,Z^J$ which are valid in the quantum theory.

It is instructive to consider the contractions of Z as well as Y and W (3.32) (again suppressing the gauge group indices)

$$\begin{aligned}
\langle Z(w_1,\bar{w}_1)\,Z(w_2,\bar{w}_2)\rangle &= \frac{1}{2\pi^2}\frac{\bar{w}_{12}}{w_{12}}\,, & \langle Y(w_1,\bar{w}_1)\,Y(w_2,\bar{w}_2)\rangle &= -\frac{1}{4\pi^2}\frac{1}{w_{12}\,\bar{w}_{12}}\,,\\
\langle Y(w_1,\bar{w}_1)\,Z(w_2,\bar{w}_2)\rangle &= \frac{1}{2\pi^2}\frac{1}{w_{12}}\,, & \langle W(w_1,\bar{w}_1)\,Z(w_2,\bar{w}_2)\rangle &= \frac{1}{2\pi^2}\frac{1}{w_{12}\,\bar{w}_{12}}\,,\\
\langle W(w_1,\bar{w}_1)\,Y(w_2,\bar{w}_2)\rangle &= 0\,, & \langle W(w_1,\bar{w}_1)\,W(w_2,\bar{w}_2)\rangle &= 0\,.
\end{aligned} \tag{3.40}$$

Consider the action of $\mathcal{Q}_a^+$ on the correlator of any number of $\mathrm{Tr}\,Z^J$ operators and one arbitrary local operator $\mathcal{O}$

$$\mathcal{Q}_a^+\big\langle\mathcal{O}\ \mathrm{Tr}\,Z^{J_2}\cdots\mathrm{Tr}\,Z^{J_n}\cdots\big\rangle = \big\langle\mathcal{Q}_a^+\mathcal{O}\ \mathrm{Tr}\,Z^{J_2}\cdots\mathrm{Tr}\,Z^{J_n}\cdots\big\rangle \tag{3.41}$$

$\mathcal{Q}_a^+$ commutes with all the Z's and the overall expression vanishes, by a Ward-Takahashi identity.

Now take $\mathcal{O} = \frac{1}{2J_1}\,Q_2{}^{\dot{2}a}\mathrm{Tr}\,Z^{J_1}$. Since we saw (3.39) that $2\hat{L}_1 = \big\{\mathcal{Q}_a^+\,, -iQ_2{}^{\dot{2}a}\big\}$ and it commutes with the Z's, we have

$$-\,i\mathcal{Q}_a^+\mathcal{O} = \frac{1}{J_1}\,\hat{L}_1\mathrm{Tr}\,Z^{J_1} = \mathrm{Tr}\,\big[(\partial_{\bar{w}}Z - Y)Z^{J_1-1}\big]\,. \tag{3.42}$$

Thus we find the following relation for the four-point function with one Y insertion

$$\big\langle\mathrm{Tr}\,[Y Z^{J_1-1}]\,\mathrm{Tr}\,Z^{J_2}\,\mathrm{Tr}\,Z^{J_3}\,\mathrm{Tr}\,Z^{J_4}\big\rangle = \frac{1}{J_1}\partial_{\bar{w}_1}\big\langle\mathrm{Tr}\,Z^{J_1}\,\mathrm{Tr}\,Z^{J_2}\,\mathrm{Tr}\,Z^{J_3}\,\mathrm{Tr}\,Z^{J_4}\big\rangle\,. \tag{3.43}$$

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As we have proven in section 3.2, the four-point function on the right-hand side is given by the free contractions of the different Z's and from this we derived an *exact* expression for the correlator on the left-hand side as well. Similar statements would hold for higher n-point functions if indeed these are not renormalized either.

To illustrate this type of relation in a particularly simple example, for the two-point function we know from (3.40) that

$$\langle Z(w_1,\bar{w}_1)\, Z(w_2,\bar{w}_2)\rangle = \frac{1}{2\pi^2}\frac{\bar{w}_{12}}{w_{12}}\,, \qquad \langle Y(w_1,\bar{w}_1)\, Z(w_2,\bar{w}_2)\rangle = \frac{1}{2\pi^2}\frac{1}{w_{12}}\,, \tag{3.44}$$

and indeed $\langle Y(w_1,\bar{w}_1)\, Z(w_2,\bar{w}_2)\rangle = \partial_{\bar{w}_1}\langle Z(w_1,\bar{w}_1)\, Z(w_2,\bar{w}_2)\rangle\,.$

The twisted symmetry generators $\hat{L}_i$ can be used to derive more such relations between correlation functions.

4 Discussion

In this paper we introduced the notion of "superprotected n-point function", the correlation function of operators all sharing supersymmetries. We focused on two main examples: In section 2 operators constructed of all six scalars and at general position in $\mathbb{R}^4$, and in section 3 operators constructed out of three real scalars and restricted to a plane.

The operators have explicit spatial dependence and in the example of section 2 this renders their tree-level correlation functions space-independent. Thus these correlation functions are given by a zero-dimensional Gaussian matrix model. Furthermore we provided different evidence for the absence of perturbative corrections to these observables. The most elegant argument is that given by de Medeiros et al. [22], who showed that the $\mathcal{N}=4$ action is exact under the supersymmetries that annihilate these operators, up to instanton terms. Beyond this somewhat formal argument we checked this cancellation using an explicit expression for the one-loop correction to all n-point functions of chiral primary operators, published in an accompanying paper [21]. In addition we relied on the general structure of the four-point function of chiral primary operators [18, 19] which implies the all-loop cancelation of quantum corrections, even including instantons [25].

The operators in section 3 have a different spatial dependance and consequently more complicated correlation functions. The free contractions are those of a two-dimensional CFT with matrix fields of dimension $(\frac{1}{2},-\frac{1}{2})$. Again, we checked the quantum corrections in a variety of ways, the all-loop corrections to all four point functions and the explicit one-loop correction to some five-point functions and one six-point function. In all these cases the quantum corrections vanished, leading one to believe that again these n-point functions are given by this free theory.

There exists another class of n-point functions that do not receive quantum corrections, the extremal correlators [26]. These correlation functions are such that the weights of the operators allow only very simple Feynman diagrams to contribute and exclude quantum corrections. Our constructions are based on a very different principle; the weights are completely arbitrary, but the type of operator is correlated with its space-time position. The simplicity is a consequence of the supersymmetry shared by all the operators.

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It would clearly be desirable to have rigorous proofs that none of the correlation functions studied in this paper receive perturbative corrections. For the case discussed in section 2, this is done in [22] by showing that the action is $\mathcal{Q}$ exact, up to instanton terms. It would be interesting to try to show the same for the operators in section 3. Furthermore, since the proof applies only to the perturbative series, it suggests that there could be instanton corrections to the n-point functions and perhaps they are also computable.

Another question we have not touched on is regarding the string duals of these n-point functions. Four-point functions have been calculated in $AdS_5 \times S^5$ [13, 14, 27] and it is known that the result is also proportional to the universal polynomial prefactor $\mathcal{R}(s, t; \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ (2.30) of [18, 19]. Hence the quantum corrections in string theory also cancel for the four-point functions. It would still be nice to have explicit calculations for our examples, since they most likely are much simpler than a generic four-point function calculation (which is quite complicated). Is there some way to organize the calculation which brings out the fact that the full result localizes to free graphs? Furthermore, would it be possible to calculate in AdS higher-point functions for these operators?

Going beyond the specific examples studied in this paper, one could ask the same question regarding any n-point function where all the operators share some supersymmetries. Are all such correlation functions protected? There are many examples where BPS Wilson loop operators are [28–36]. Likewise the known examples of correlation functions of local operators and Wilson loops sharing some supersymmetries are given by summing free propagators [37–43]. One could also find Wilson loops that share supersymmetry with the n-point functions discussed in this paper [23]. As another example, the slightly more exotic surface operators [44] seem to have very simple correlation functions with Wilson loops and local operators when they all share supersymmetry [45, 46].

In fact, another family of local operators that share supersymmetry can be derived from taking infinitesimal Wilson loops. There are two known examples of families of Wilson loop operators which all share some supercharges [47, 48]. While those papers concentrate on the expectation value of a single Wilson loop, there is no impediment to take more than one — that configuration is still supersymmetric. When shrinking all the Wilson loops to small size, one ends up with local operators which can serve as another realization of the ideas put forth in this paper. The Wilson loops are made by including special scalar couplings in addition to the gauge connection. The resulting local operators will include the field strength, derivatives and commutators of scalar fields, which can all be represented in terms of some modified covariant derivative like

$$\begin{aligned} \widehat{\mathcal{D}}_\mu &= \partial_\mu - iA_\mu + \Phi_\mu \,, \\ \widetilde{\mathcal{D}}_\mu &= \partial_\mu - iA_\mu + \Phi^+_{\mu\nu} x^\nu \,. \end{aligned} \tag{4.1}$$

These correspond to the two examples, where the scalar fields get assigned space-time indices of a vector and self-dual tensor in a natural way [47, 48].

In the case of the loops constructed by Zarembo [47], the expectation values are always unity [49–51], and it is reasonable to expect that this would be true also in the limit. The second example, that in [48] is more complicated and one would expect the n-point function of the infinitesimal Wilson loops to be non-zero. In particular, when the loop is restricted to an S^2 in space-time there is some evidence showing that they are equal to a perturbative calculation in two-dimensional Yang-Mills theory [52, 53] (see also [54, 55]). Are the correlation functions of the infinitesimal loops then given by the correlators of single plaquette operators in two-dimensional Yang-Mils?

These examples, including the ones studied in this paper are surely not the only ones. For local operators, as mentioned before, any three chiral primary operators will share some supercharges. It is reasonable to expect that on the line (or circle) spanned by these operators one could place more local operators that share the same supercharges as the original three. Will all such objects have vanishing quantum corrections? It is possible that the tools we used in section 3 would apply also there. In checking for the cancelation of the one-loop corrections an important property was the way the interaction vertex depended on the complex cross-ratio (3.27). For four operators on a line there is only one real cross-ratio, so it is possible that similar relations will also hold.

Beyond a single line, one can ask whether there are other examples of families of operators on submanifolds of space-time, like $\mathbb{R}^2$ in our second example, that share some supercharges, and whether they receive quantum corrections. One useful tool may be the universal polynomial function $\mathcal{R}$ (2.30). In both of our examples it vanished, proving that the four point functions do not get renormalized. One can therefore ask for which collection of points, or submanifold of space-time and for which operators does $\mathcal{R}$ vanish. In these cases will the operators necessarily share some supercharges? Under what conditions would it be possible to add operators to make n-point functions with vanishing quantum corrections, and is there a generalization of $\mathcal{R}$ to these cases (see also [21]).

As we touched on in the text, the supersymmetry shared by the families of operators we constructed lead to some bosonic "twisted" symmetries that relate different correlation functions to each-other. It would be interesting to understand the scope of these symmetries and find all possible correlation functions of other operators, involving fermions, derivatives and gauge fields and which are related to the ones we have calculated — and therefore are also "superprotected".

Our results advocate the point of view where one should not necessarily regard local operators as the basic objects and n-point functions merely as their correlators. The n-point functions may have more of an independent meaning. One example of this dual point of view are classical geodesics in AdS space — they calculate the two-point functions of dual operators. In particular, in all the examples that we studied we investigated the amount of supersymmetry preserved by all the objects in the correlation function, not each separately.

A very interesting spin-off would be to try to build upon our "superprotected" three-point functions to understand the interaction of non-BPS operators. In the same way that the spectrum of local operators is understood in terms of magnon excitations over a supersymmetric ground state, one could put magnons on top of three long operators which share supersymmetry and study their interactions. We find the operators in section 2 particularly promising candidates for the ground state, since their correlation functions have trivial spatial dependence.

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A Notations and the superalgebra

This appendix summarizes our conventions for the $\mathcal{N} = 4$ superconformal algebra PSU(2, 2|4) following [56]. The two ways of breaking the R-symmetry group SU(4) $\to$ SU(2) $\times$ SU(2) are then explained in the following appendices.

We denote by $J^\alpha{}_\beta$, $\bar{J}^{\dot\alpha}{}_{\dot\beta}$ the generators of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ Lorentz group, and by $R^A{}_B$ the 15 generators of the R-symmetry group SU(4). The remaining bosonic generators are the translations $P_{\alpha\dot\alpha}$, the special conformal transformations $K^{\alpha\dot\alpha}$ and the dilatation D. Finally the 32 fermionic generators are the Poincaré supersymmetries Q^A_α, $\bar{Q}_{\dot\alpha A}$ and the superconformal supersymmetries S^α_A, $\bar{S}^{\dot\alpha A}$.

The commutators of any generator $\mathcal{G}$ with $J^\alpha{}_\beta$, $\bar{J}^{\dot\alpha}{}_{\dot\beta}$ and $R^A{}_B$ are canonically dictated by the index structure

$$\left[J^\alpha{}_\beta\,,\mathcal{G}_\gamma\right] = \delta^\alpha_\gamma\mathcal{G}_\beta - \frac{1}{2}\delta^\alpha_\beta\mathcal{G}_\gamma\,, \qquad \left[J^\alpha{}_\beta\,,\mathcal{G}^\gamma\right] = -\delta^\gamma_\beta\mathcal{G}^\alpha + \frac{1}{2}\delta^\alpha_\beta\mathcal{G}^\gamma\,, \tag{A.1}$$

$$\left[\bar{J}^{\dot\alpha}{}_{\dot\beta}\,,\mathcal{G}_{\dot\gamma}\right] = \delta^{\dot\alpha}_{\dot\gamma}\mathcal{G}_{\dot\beta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathcal{G}_{\dot\gamma}\,, \qquad \left[\bar{J}^{\dot\alpha}{}_{\dot\beta}\,,\mathcal{G}^{\dot\gamma}\right] = -\delta^{\dot\gamma}_{\dot\beta}\mathcal{G}^{\dot\alpha} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathcal{G}^{\dot\gamma} \tag{A.2}$$

$$\left[R^A{}_B\,,\mathcal{G}_C\right] = \delta^A_C\mathcal{G}_B - \frac{1}{4}\delta^A_B\mathcal{G}_C\,, \qquad \left[R^A{}_B\,,\mathcal{G}^C\right] = -\delta^C_B\mathcal{G}^A + \frac{1}{4}\delta^A_B\mathcal{G}^C\,. \tag{A.3}$$

while commutators with the dilatation operator D are given by $\left[D\,,\mathcal{G}\right] = \dim(\mathcal{G})\,\mathcal{G}$, where $\dim(\mathcal{G})$ is the dimension of the generator $\mathcal{G}$.

The remaining non-trivial commutators are

$$\begin{aligned}
\left\{Q^A_\alpha\,,\bar{Q}_{\dot\alpha B}\right\} &= \delta^A_B P_{\alpha\dot\alpha}\,, & \left\{S^\alpha_A\,,\bar{S}^{\dot\alpha B}\right\} &= \delta^B_A K^{\alpha\dot\alpha}\,,\\
\left[K^{\alpha\dot\alpha}\,,Q^A_\beta\right] &= \delta^\alpha_\beta\bar{S}^{\dot\alpha A}\,, & \left[K^{\alpha\dot\alpha}\,,\bar{Q}_{\dot\beta A}\right] &= \delta^{\dot\alpha}_{\dot\beta}S^\alpha_A\,,\\
\left[P_{\alpha\dot\alpha}\,,S^\beta_A\right] &= -\delta^\beta_\alpha\bar{Q}_{\dot\alpha A}\,, & \left[P_{\alpha\dot\alpha}\,,\bar{S}^{\dot\beta A}\right] &= -\delta^{\dot\beta}_{\dot\alpha}Q^A_\alpha\,,\\
\left\{Q^A_\alpha\,,S^\beta_B\right\} &= \delta^A_B J^\beta{}_\alpha + \delta^\beta_\alpha R^A{}_B + \frac{1}{2}\delta^A_B\delta^\beta_\alpha D\,,\\
\left\{\bar{Q}_{\dot\alpha A}\,,\bar{S}^{\dot\beta B}\right\} &= \delta^B_A\bar{J}^{\dot\beta}{}_{\dot\alpha} - \delta^{\dot\beta}_{\dot\alpha}R^B{}_A + \frac{1}{2}\delta^B_A\delta^{\dot\beta}_{\dot\alpha}D\,,\\
\left[K^{\alpha\dot\alpha}\,,P_{\beta\dot\beta}\right] &= \delta^{\dot\alpha}_{\dot\beta}J^\alpha{}_\beta + \delta^\alpha_\beta\bar{J}^{\dot\alpha}{}_{\dot\beta} + \delta^\alpha_\beta\delta^{\dot\alpha}_{\dot\beta}D\,.
\end{aligned} \tag{A.4}$$

So far we have written the algebra in spinor notations, but we find it useful also to transform to vector notations. To that end we take the following choice of Euclidean gamma matrices for $\mathbb{R}^4$, where τ^i are the usual Pauli matrices

$$\gamma^i = \begin{pmatrix} 0 & (\sigma_i)_{\alpha\dot\alpha} \\ (\bar\sigma^i)^{\dot\alpha\alpha} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\tau^i \\ -i\tau^i & 0 \end{pmatrix} \qquad \gamma^4 = \begin{pmatrix} 0 & (\sigma_4)_{\alpha\dot\alpha} \\ (\bar\sigma^4)^{\dot\alpha\alpha} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{A.5}$$

SU(2) indices can be raised and lowered by using the appropriate epsilon tensor, for which we adopt the conventions

$$\mathcal{G}^r = \varepsilon^{rs}\mathcal{G}_s\,, \quad \mathcal{G}_r = \varepsilon_{rs}\mathcal{G}^s\,; \qquad \varepsilon^{rs} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \quad \varepsilon_{rs} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{A.6}$$

where the indices r, s belong to any SU(2) group. Indeed $(\bar\sigma^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\,\epsilon^{\alpha\beta}\,(\sigma^\mu)_{\beta\dot\beta}$.

We note the contraction relations

$$(\sigma^\mu)_{\alpha\dot\alpha}\,(\bar\sigma^\mu)^{\dot\beta\beta} = 2\,\delta^{\dot\beta}_{\dot\alpha}\,\delta^\beta_\alpha \qquad \mathrm{Tr}\,(\sigma^\mu\bar\sigma^\nu) = 2\,\delta^{\mu\nu}\,. \tag{A.7}$$

The gamma matrices with anti-symmetric indices are

$$\sigma^{\mu\nu} = \frac{1}{2}\,(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\,, \qquad \bar\sigma^{\mu\nu} = \frac{1}{2}\,(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)\,. \tag{A.8}$$

We may now define

$$\begin{aligned} P^\mu &= P_{\alpha\dot\alpha}\,(\bar\sigma^\mu)^{\dot\alpha\alpha}\,, & P_{\dot\alpha\alpha} &= \frac{1}{2}\,(\sigma^\mu)_{\alpha\dot\alpha}\,P_\mu\,, \\ K^\mu &= K^{\dot\alpha\alpha}\,(\sigma^\mu)_{\alpha\dot\alpha}\,, & K^{\dot\alpha\alpha} &= \frac{1}{2}\,(\bar\sigma_\mu)^{\dot\alpha\alpha}\,K^\mu\,, \\ J^{\mu\nu} &= \frac{1}{2}\left(J^\alpha{}_\beta\,\sigma^{\mu\nu})_\alpha{}^\beta - \bar J^{\dot\alpha}{}_{\dot\beta}\,(\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\right). \end{aligned} \tag{A.9}$$

Using the commutation relations (A.4) and contracting the relevant σ^μ and $\bar\sigma^\nu$ we get the commutators in SO(4) language

$$\begin{aligned} \left[K^\mu,\, P^\nu\right] &= 2\,(J^{\mu\nu} + \delta^{\mu\nu}\,D) \\ \left[J^{\mu\nu},\, P^\rho\right] &= \delta^{\mu\rho}\,P^\nu - \delta^{\nu\rho}\,P^\mu\,, \\ \left[J^{\mu\nu},\, J^{\rho\sigma}\right] &= \delta^{\mu\rho}\,J^{\nu\sigma} - \delta^{\nu\rho}\,J^{\mu\sigma} + \delta^{\mu\sigma}\,J^{\rho\nu} - \delta^{\nu\sigma}\,J^{\rho\mu}\,. \end{aligned} \tag{A.10}$$

These commutation relations can be realized by the following definition of the action of the symmetry generators on scalar fields

$$\begin{aligned} P_\mu\,\Phi^i &= \partial_\mu\Phi^i\,, \\ J_{\mu\nu}\,\Phi^i &= (x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi^i\,, \\ D\,\Phi^i &= (x^\mu\partial_\mu + \Delta)\Phi^i\,, \\ K_\mu\,\Phi^i &= (2x_\mu x^\nu\partial_\nu + 2\Delta x_\mu - x^2\partial_\mu)\Phi^i\,. \end{aligned} \tag{A.11}$$

Noting that to calculate the commutators the derivatives act on fields, and not directly on the coordinates, one gets the commutation relations (A.10).

The action of the R-symmetry generators on the scalar fields can be written as

$$R_{ij}\,\Phi^k = \delta_i^k \Phi_j - \delta_j^k \Phi_i\,, \tag{A.12}$$

which gives the algebra

$$\big[R_{ij}\,, R_{kl}\big] = \delta_{ik}R_{jl} - \delta_{jk}R_{li} + \delta_{il}R_{kj} - \delta_{jl}R_{ik}\,, \tag{A.13}$$

We choose specific notations for the R-symmetry generators in the following two appendices, once we break SO(6) to SU(2) $\times$ SU(2) in the two ways appropriate for the different local operators discussed in the text.

B Symmetry breaking for example I

The construction of the operators in section 2 involves an identification of the full SO(5,1) conformal group and the R-symmetry group. The supercharges, which transform in two bi-spinor representations of those groups may be decomposed, after the identification, to two adjoints and two singlets of the diagonal group. The supercharges preserved by the field C are the singlets. The standard notations have the SO(4) Euclidean Lorentz group written as $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ with the spinors in the $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$ representations, labeled by the indices α and $\dot\alpha$. Therefore, to describe the supersymmetry preserved by the operators on $\mathbb{R}^4$ it is useful to consider the breaking of the SU(4) R-symmetry group to $\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ such that the spinor representation becomes $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$. We will use indices $\dot a$ for $\mathrm{SU}(2)_A$ and a for $\mathrm{SU}(2)_B$. Note that a different breaking is used for the operators on $\mathbb{R}^2$ and will be described below in appendix C.

Under this breaking the supergroup generators

$$\left(\begin{array}{cc|c} \begin{matrix} J_\alpha{}^\beta + \frac12\delta_\alpha^\beta D \\ -K^{\dot\alpha\beta}\end{matrix} & \begin{matrix} P_{\alpha\dot\beta} \\ -\bar J^{\dot\alpha}{}_{\dot\beta} - \frac12\delta^{\dot\alpha}_{\dot\beta} D\end{matrix} & \begin{matrix} Q_\alpha^B \\ -\bar S^{\dot\alpha B}\end{matrix} \\ \hline S_A^\beta & \bar Q_{\dot\beta A} & R^B{}_A \end{array}\right) \tag{B.1}$$

are decomposed as

$$\left(\begin{array}{cc|cc} \begin{matrix} J_\alpha{}^\beta + \frac12\delta_\alpha^\beta D \\ -K^{\dot\alpha\beta}\end{matrix} & \begin{matrix} P_{\alpha\dot\beta} \\ -\bar J^{\dot\alpha}{}_{\dot\beta} - \frac12\delta^{\dot\alpha}_{\dot\beta} D\end{matrix} & \begin{matrix} Q_\alpha^b \\ -\bar S^{\dot\alpha b}\end{matrix} & \begin{matrix} \dot Q_{\alpha\dot b} \\ -\dot{\bar S}{}^{\dot\alpha}_{\dot b}\end{matrix} \\ \hline \begin{matrix} S_a^\beta \\ -\dot S^{\beta\dot a}\end{matrix} & \begin{matrix} \bar Q_{\dot\beta a} \\ -\dot{\bar Q}{}^{\dot a}_{\dot\beta}\end{matrix} & \begin{matrix} R^b{}_a + \frac12\delta_a^b \dot D \\ -\dot K^{\dot a b}\end{matrix} & \begin{matrix} \dot P_{a\dot b} \\ -\dot R^{\dot a}{}_{\dot b} - \frac12\delta^{\dot a}_{\dot b}\dot D\end{matrix} \end{array}\right) \tag{B.2}$$

This decomposition of the PSU(2,2|4) algebra into $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ is realized in a very simple way using the oscillator picture of [57]. One starts with two pairs of bosonic oscillators $(\alpha, \dot\alpha = 1, 2)$

$$\big[a^\alpha,\, a^\dagger_\beta\big] = \delta^\alpha_\beta\,, \qquad \big[b^{\dot\alpha},\, b^\dagger_{\dot\beta}\big] = \delta^{\dot\alpha}_{\dot\beta}\,, \tag{B.3}$$

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and four fermionic oscillators ($A = 1, 2, 3, 4$)

$$\{c^A,\, c^\dagger_B\} = \delta^A_B\,. \tag{B.4}$$

Then one rewrites the fermionic generators in terms of the two pairs c^a and $d^{\dot{a}}$ (with $a, \dot{a} = 1, 2$ and standard anti-commutators)

$$c^A = (\, c^1, c^2, d^\dagger_{\dot{1}}, d^\dagger_{\dot{2}}\,) \qquad c^\dagger_A = (\, c^\dagger_1, c^\dagger_2, d^{\dot{1}}, d^{\dot{2}}\,) \tag{B.5}$$

The bosonic generators of the algebra are made either of two bosonic oscillators (giving the conformal part) or two fermionic ones (giving the R-symmetry part)

$$\begin{aligned}
&J^\alpha{}_\beta = a^\dagger_\beta\, a^\alpha - \frac{1}{2}\delta^\alpha_\beta\, a^\dagger_\gamma\, a^\gamma && \bar{J}^{\dot\alpha}{}_{\dot\beta} = b^\dagger_{\dot\beta}\, b^{\dot\alpha} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\, b^\dagger_{\dot\gamma}\, b^{\dot\gamma} && \\
&P_{\alpha\dot\beta} = a^\dagger_\alpha\, b^\dagger_{\dot\beta} && K^{\alpha\dot\beta} = a^\alpha\, b^{\dot\beta} && D = 1 + \frac{1}{2}(a^\dagger_\gamma a^\gamma + b^\dagger_{\dot\gamma} b^{\dot\gamma}) \\
&R^a{}_b = c^\dagger_b\, c^a - \frac{1}{2}\delta^a_b\, c^\dagger_c\, c^c && \dot{R}^{\dot a}{}_{\dot b} = d^\dagger_{\dot b}\, d^{\dot a} - \frac{1}{2}\delta^{\dot a}_{\dot b}\, d^\dagger_{\dot c}\, d^{\dot c} && \\
&\dot{P}_{a\dot b} = c^\dagger_a\, d^\dagger_{\dot b} && \dot{K}^{a\dot b} = c^a\, d^{\dot b} && \dot{D} = -1 + \frac{1}{2}(c^\dagger_c c^c + d^\dagger_{\dot c} d^{\dot c})
\end{aligned} \tag{B.6}$$

The fermionic generators of the superalgebra can then be written as

$$\begin{aligned}
&Q^a{}_\alpha = a^\dagger_\alpha\, c^a\,, && \bar{Q}_{a\dot\alpha} = b^\dagger_{\dot\alpha}\, c^\dagger_a\,, && S^\alpha{}_a = c^\dagger_a\, a^\alpha\,, && \bar{S}^{\dot\alpha a} = b^{\dot\alpha}\, c^a\,, \\
&\dot{Q}_{\dot a\alpha} = a^\dagger_\alpha\, d^\dagger_{\dot a}\,, && \dot{\bar{Q}}^{\dot a}{}_{\dot\alpha} = -b^\dagger_{\dot\alpha}\, d^{\dot a}\,, && \dot{S}^{\alpha\dot a} = -a^\alpha\, d^{\dot a}\,, && \dot{\bar{S}}^{\dot\alpha}{}_{\dot a} = d^\dagger_{\dot a}\, b^{\dot\alpha}\,.
\end{aligned} \tag{B.7}$$

Some of their commutators are

$$\begin{aligned}
&\{Q^a_\alpha,\, \bar{Q}_{b\dot\alpha}\} = \delta^a_b P_{\alpha\dot\alpha}, && \{\dot{Q}_{\alpha\dot a},\, \dot{\bar{Q}}^{\dot b}_{\dot\alpha}\} = -\delta^{\dot b}_{\dot a} P_{\alpha\dot\alpha}\,, \\
&\{S^\alpha_a,\, \bar{S}^{\dot\alpha b}\} = \delta^b_a K^{\alpha\dot\alpha}\,, && \{\dot{S}^{\alpha\dot a},\, \dot{\bar{S}}^{\dot\alpha}_{\dot b}\} = -\delta^{\dot a}_{\dot b} K^{\alpha\dot\alpha}\,, \\
&[K^{\alpha\dot\alpha},\, Q^a_\beta] = \delta^\alpha_\beta \bar{S}^{\dot\alpha a}\,, && [K^{\alpha\dot\alpha},\, \dot{\bar{Q}}^{\dot a}_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta} \dot{S}^{\alpha\dot a}\,, \\
&[P_{\alpha\dot\alpha},\, S^\beta_a] = -\delta^\beta_\alpha \bar{Q}_{\dot\alpha a}\,, && [P_{\alpha\dot\alpha},\, \dot{\bar{S}}^{\dot\beta}_{\dot a}] = -\delta^{\dot\beta}_{\dot\alpha} \dot{Q}_{\alpha\dot a}\,, \\
&\{Q^a_\alpha,\, S^\beta_b\} = \delta^a_b J^\beta{}_\alpha + \delta^\beta_\alpha R^a{}_b + \frac{1}{2}\delta^a_b \delta^\beta_\alpha (D + \dot{D})\,, && \{Q^a_\alpha,\, \dot{S}^{\beta\dot b}\} = \delta^\beta_\alpha \dot{K}^{\dot b a}\,, \\
&\{\dot{Q}_{\dot a\alpha},\, \dot{S}^{\beta\dot b}\} = -\delta^{\dot b}_{\dot a} J^\beta{}_\alpha + \delta^\beta_\alpha \dot{R}^{\dot b}{}_{\dot a} - \frac{1}{2}\delta^{\dot b}_{\dot a} \delta^\beta_\alpha (D - \dot{D})\,, && \{\dot{Q}_{\dot a\alpha},\, S^\beta_b\} = \delta^\beta_\alpha \dot{P}_{b\dot a}\,, \\
&\{\bar{Q}_{\dot\alpha a},\, \bar{S}^{\dot\beta b}\} = -\delta^b_a \bar{J}^{\dot\beta}{}_{\dot\alpha} + \delta^{\dot\beta}_{\dot\alpha} R^b{}_a - \frac{1}{2}\delta^b_a \delta^{\dot\beta}_{\dot\alpha} (D - \dot{D})\,, && \{\bar{Q}_{\dot\alpha a},\, \dot{\bar{S}}^{\dot\beta}_{\dot b}\} = -\delta^{\dot\beta}_{\dot\alpha} \dot{P}_{a\dot b}\,, \\
&\{\dot{\bar{Q}}^{\dot a}_{\dot\alpha},\, \dot{\bar{S}}^{\dot\beta}_{\dot b}\} = -\delta^{\dot a}_{\dot b} \bar{J}^{\dot\beta}{}_{\dot\alpha} - \delta^{\dot\beta}_{\dot\alpha} \dot{R}^{\dot a}{}_{\dot b} - \frac{1}{2}\delta^{\dot a}_{\dot b} \delta^{\dot\beta}_{\dot\alpha} (D + \dot{D})\,, && \{\dot{\bar{Q}}^{\dot a}_{\dot\alpha},\, \bar{S}^{\dot\beta b}\} = -\delta^{\dot\beta}_{\dot\alpha} \dot{K}^{\dot a b}\,.
\end{aligned} \tag{B.8}$$

The construction of the field C in section 2 involves an identification between the conformal group and the R-symmetry group. In particular this gives a canonical identification between the undotted indices of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_B$ and between the dotted ones of $\mathrm{SU}(2)_R$ and $\mathrm{SU}(2)_A$. This allows one to define the traced supersymmetry generators

$$\begin{aligned}
&Q = Q^\alpha{}_\alpha = a^\dagger_\alpha\, c^\alpha\,, && \dot{\bar{Q}} = \dot{\bar{Q}}^{\dot\alpha}{}_{\dot\alpha} = b^\dagger_{\dot\alpha}\, d^{\dot\alpha}\,, \\
&S = S^\alpha{}_\alpha = c^\dagger_\alpha\, a^\alpha\,, && \dot{\bar{S}} = \dot{\bar{S}}^{\dot\alpha}{}_{\dot\alpha} = d^\dagger_{\dot\alpha}\, b^{\dot\alpha}\,.
\end{aligned} \tag{B.9}$$

These generators are invariant under the diagonal sums of the SU(2) factors, but not over the full sum of the conformal group and R-symmetry group. The two generators that are invariant under that identification require fully tracing over the off-diagonal blocks in (B.2). The resulting two supercharges which anti-commute with each-other are

$$\mathcal{Q}^+ = Q - \dot{\bar{S}}\,, \qquad \mathcal{Q}^- = \dot{\bar{Q}} - S\,. \tag{B.10}$$

Under this identification it is also possible to assign space-time indices to the R-symmetry generators and to the remaining supercharges. Using the usual γ matrices (now with $a, \dot{a}$ indices) we have

$$\begin{aligned} \dot{P}_\mu &= \dot{P}_{a\dot{a}}\,(\bar{\sigma}_\mu)^{\dot{a}a} = R_{5\mu} + iR_{6\mu}\,, & \dot{K}_\mu &= \dot{K}^{\dot{a}a}\,(\sigma_\mu)_{a\dot{a}} = R_{5\mu} - iR_{6\mu}\,, \\ R_{\mu\nu} &= \frac{1}{2}\left(R^a{}_b\,(\sigma_{\mu\nu})_a{}^b - \dot{R}^{\dot{a}}{}_{\dot{b}}\,(\bar{\sigma}_{\mu\nu})^{\dot{b}}{}_{\dot{a}}\right), & \dot{D} &= iR_{56}\,. \end{aligned} \tag{B.11}$$

For the supercharges we take the combinations

$$\begin{aligned} Q_\mu &= (\bar{\sigma}_\mu)^{\dot{a}\alpha}(\bar{Q}_{\alpha\dot{a}} - \dot{Q}_{\alpha\dot{a}}) \qquad\qquad S_\mu = (\sigma_\mu)_{a\dot{\alpha}}(\dot{S}^{\dot{\alpha}a} - \bar{S}^{\dot{\alpha}a}) \\ Q_{\mu\nu} &= \frac{1}{2}\left((S^a_\alpha - Q^a_\alpha)(\sigma_{\mu\nu})_a{}^\alpha - (\dot{\bar{Q}}^{\dot{\alpha}}_{\dot{a}} - \dot{\bar{S}}^{\dot{\alpha}}_{\dot{a}})(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{a}}\right) \\ Q_D &= \frac{1}{2}\left(S^a_a - Q^a_a + \dot{\bar{Q}}^{\dot{a}}_{\dot{a}} - \dot{\bar{S}}^{\dot{a}}_{\dot{a}}\right) \end{aligned} \tag{B.12}$$

Acting on them with $\mathcal{Q}^\pm$ gives the twisted generators (2.15) which are the sum of the conformal generators (A.9) and the R-symmetries (B.11)

$$\begin{aligned} \left\{\mathcal{Q}^\pm, Q_\mu\right\} &= \hat{P}_\mu = P_\mu + \dot{P}_\mu\,, & \left\{\mathcal{Q}^\pm, Q_{\mu\nu}\right\} &= \hat{J}_{\mu\nu} = J_{\mu\nu} + R_{\mu\nu}\,, \\ \left\{\mathcal{Q}^\pm, S_\mu\right\} &= \hat{K}_\mu = K_\mu + \dot{K}_\mu\,, & \left\{\mathcal{Q}^\pm, Q_D\right\} &= \hat{D} = D + \dot{D}\,. \end{aligned} \tag{B.13}$$

Under the action of these generators the field C transforms as a dimension-zero scalar (2.16).

We would like to comment that after choosing the scalar field C (2.1), it is natural to arrange the five other scalar fields as [22]

$$\begin{aligned} V^\mu &= i\Phi^\mu + x^\mu(\Phi^6 - i\Phi^5)\,, \\ B &= \Phi^6 - i\Phi^5\,. \end{aligned} \tag{B.14}$$

The full twisted conformal group (2.15) as well as the twisted supercharges (B.12) give many more relations among the correlation functions of operators made of C and these fields. For example the twisted conformal generators acting on V^μ give

$$\begin{aligned} \hat{P}_\mu\,V^\nu &= \partial_\mu V^\nu\,, \\ \hat{J}_{\mu\nu}\,V^\rho &= (x_\mu\partial_\nu - x_\nu\partial_\mu)V^\rho + \delta^\rho_\mu V_\nu - \delta^\rho_\nu V_\mu\,, \\ \hat{D}\,V^\mu &= x^\nu\partial_\nu V^\mu + V^\mu\,, \\ \hat{K}_\mu\,V^\nu &= (2x_\mu x^\nu\partial_\nu + 2x_\mu - x^2\partial_\mu)V^\nu - 2x_\mu V^\nu + \delta^\nu_\mu(2x_\rho V^\rho - C)\,. \end{aligned} \tag{B.15}$$

and the action on B is

$$\begin{aligned}\hat{P}_\mu\, B &= \partial_\mu B\,,\\ \hat{J}_{\mu\nu}\, B &= (x_\mu\partial_\nu - x_\nu\partial_\mu)B\,,\\ \hat{D}\, B &= x^\mu\partial_\mu B + 2B\,,\\ \hat{K}_\mu\, B &= (2x_\mu x^\nu\partial_\nu + 4x_\mu - x^2\partial_\mu)B - 2V_\mu\,.\end{aligned} \tag{B.16}$$

We will not explore further the consequences of these relations here.

C Symmetry breaking for example II

The construction of the field C on $\mathbb{R}^2$ involves choosing three of the real scalars, so it explicitly breaks the R-symmetry group $\mathrm{SU}(4) \to \mathrm{SU}(2)_{A'} \times \mathrm{SU}(2)_{B'}$. Unlike the breaking in appendix B, here the breaking is such that the $\mathbf{4}$ of SU(4) becomes the $(\mathbf{2},\,\mathbf{2})$ of $\mathrm{SU}(2)_{A'} \times \mathrm{SU}(2)_{B'}$. Now the supercharges will carry indices of both groups, a dotted one for $\mathrm{SU}(2)_{A'}$ and an undotted one for $\mathrm{SU}(2)_{B'}$.

This breaking of $\mathrm{SU}(4) \to \mathrm{SU}(2)_{A'} \times \mathrm{SU}(2)_{B'}$ is very similar to that required for the study of the supersymmetric Wilson loops of [48, 53] and much of this appendix is copied from appendix A of [53].

The R-symmetry generators decompose under $\mathrm{SU}(4) \to \mathrm{SU}(2)_{A'} \times \mathrm{SU}(2)_{B'}$ as $\mathbf{15} \to (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{3})$. This can be explicitly written as

$$R^A{}_B \to R^{\dot{a}a}{}_{\dot{b}b} = \frac{1}{2}\delta^a_b \dot{T}^{\dot{a}}{}_{\dot{b}} + \frac{1}{2}\delta^{\dot{a}}_{\dot{b}} T^a{}_b + \frac{1}{2}M^{\dot{a}a}{}_{\dot{b}b} \tag{C.1}$$

where $\dot{T}^{\dot{a}}_{\dot{b}}$ and T^a_b are respectively the $\mathrm{SU}(2)_{A'}$ and $\mathrm{SU}(2)_{B'}$ generators, and the 9 generators in the $(\mathbf{3},\mathbf{3})$ are given by $M^{\dot{a}a}{}_{\dot{b}b}$, which is traceless in each pair of indices

$$\delta^{\dot{b}}_{\dot{a}} M^{\dot{a}a}{}_{\dot{b}b} = \delta^b_a M^{\dot{a}a}{}_{\dot{b}b} = 0\,. \tag{C.2}$$

The commutation relations of the supercharges written in $\mathrm{SU}(2)_{A'} \times \mathrm{SU}(2)_{B'}$ notation are

$$\begin{aligned}
&\{Q^{\dot{a}a}_\alpha,\, \bar{Q}_{\dot{\alpha}\dot{b}b}\} = \delta^{\dot{a}}_{\dot{b}}\delta^a_b P_{\alpha\dot{\alpha}}\,, \qquad\qquad \{S^\alpha_{\dot{a}a},\, \bar{S}^{\dot{\alpha}\dot{b}b}\} = \delta^{\dot{b}}_{\dot{a}}\delta^b_a K^{\alpha\dot{\alpha}}\,,\\
&[K^{\alpha\dot{\alpha}},\, Q^{\dot{a}a}_\beta] = \delta^\alpha_\beta \bar{S}^{\dot{\alpha}\dot{a}a}\,, \qquad\qquad [K^{\alpha\dot{\alpha}},\, \bar{Q}_{\dot{\beta}\dot{a}a}] = \delta^{\dot{\alpha}}_{\dot{\beta}} S^\alpha_{\dot{a}a}\,,\\
&[P_{\alpha\dot{\alpha}},\, S^\beta_{\dot{a}a}] = \delta^\beta_\alpha \bar{Q}_{\dot{\alpha}\dot{a}a}\,, \qquad\qquad [P_{\alpha\dot{\alpha}},\, \bar{S}^{\dot{\beta}\dot{a}a}] = \delta^{\dot{\beta}}_{\dot{\alpha}} Q^{\dot{a}a}_\alpha\,,\\
&\{Q^{\dot{a}a}_\alpha,\, S^\beta_{\dot{b}b}\} = \delta^{\dot{a}}_{\dot{b}}\delta^a_b J^\beta{}_\alpha + \frac{1}{2}\delta^\beta_\alpha\left(\delta^a_b \dot{T}^{\dot{a}}{}_{\dot{b}} + \delta^{\dot{a}}_{\dot{b}} T^a{}_b + M^{\dot{a}a}{}_{\dot{b}b} + \delta^{\dot{a}}_{\dot{b}}\delta^a_b D\right)\,,\\
&\{\bar{Q}_{\dot{\alpha}\dot{a}a},\, \bar{S}^{\dot{\beta}\dot{b}b}\} = \delta^{\dot{b}}_{\dot{a}}\delta^b_a \bar{J}^{\dot{\beta}}{}_{\dot{\alpha}} - \frac{1}{2}\delta^{\dot{\beta}}_{\dot{\alpha}}\left(\delta^b_a \dot{T}^{\dot{b}}{}_{\dot{a}} + \delta^{\dot{b}}_{\dot{a}} T^b{}_a + M^{\dot{b}b}{}_{\dot{a}a} - \delta^{\dot{b}}_{\dot{a}}\delta^b_a D\right)\,.
\end{aligned} \tag{C.3}$$

In section 3 we use also the R-symmetry generators with SO(6) vector indices R_{ij}. It is useful therefore to identify them, for $i = 1,2,3$, with the rotations $\dot{T}^{\dot{a}}{}_{\dot{b}}$. This is done through

$$R_{ij} = -\frac{1}{2}(\rho_{ij})^{\dot{a}}{}_{\dot{b}}\dot{T}^{\dot{b}}{}_{\dot{a}}\,, \qquad \dot{T}^{\dot{a}}{}_{\dot{b}} = \frac{1}{2}(\rho^{ij})^{\dot{a}}{}_{\dot{b}}R_{ij}\,, \qquad (\rho_{ij})^{\dot{a}}{}_{\dot{b}} = i\,\varepsilon_{ijk}(\tau^k)^{\dot{a}}{}_{\dot{b}}\,. \tag{C.4}$$

The commutators are

$$[\dot{T}^{\dot{a}}{}_{\dot{b}},\, \dot{T}^{\dot{c}}{}_{\dot{d}}] = -\delta^{\dot{a}}_{\dot{d}}\dot{T}^{\dot{c}}{}_{\dot{b}} + \delta^{\dot{c}}_{\dot{b}}\dot{T}^{\dot{a}}{}_{\dot{d}} \quad\Leftrightarrow\quad [R_{ij},\, R_{kl}] = \delta_{ik}R_{jl} - \delta_{jk}R_{il} + \delta_{il}R_{kj} - \delta_{jl}R_{ki}\,, \tag{C.5}$$

like in (A.10).

C.1 Action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$

All operators in the plane transform in representations of the rigid conformal group $\mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The field Z (3.2) as well as Y and W (3.32) carry also $\mathrm{SU}(2)_{A'}$ indices and transform under this group. In section 3.3 we discussed the action of these generators, which we elaborate on here.

We write the holomorphic, anti-holomorphic and $\mathrm{SU}(2)_{A'}$ algebras in terms of raising and lowering operators

$$L_1 = \frac{1}{2}(P_1 - iP_2)\,, \qquad L_0 = \frac{1}{2}(D - iJ_{12})\,, \qquad L_{-1} = \frac{1}{2}(K_1 + iK_2)\,, \tag{C.6}$$

$$\bar{L}_1 = \frac{1}{2}(P_1 + iP_2)\,, \qquad \bar{L}_0 = \frac{1}{2}(D + iJ_{12})\,, \qquad \bar{L}_{-1} = \frac{1}{2}(K_1 - iK_2)\,, \tag{C.7}$$

$$R_+ = -i(R_{23} + iR_{31})\,, \qquad R_0 = iR_{12}\,, \qquad R_- = i(R_{23} - iR_{31})\,, \tag{C.8}$$

The holomorpic operators act on the fields by

$$\begin{aligned}
&L_1\, Z = \partial_w Z\,, && L_0\, Z = w\,\partial_w Z + \frac{1}{2}Z\,, && L_{-1}\, Z = w^2\,\partial_w Z + wZ\,,\\
&L_1\, Y = \partial_w Y\,, && L_0\, Y = w\,\partial_w Y + \frac{1}{2}Y\,, && L_{-1}\, Y = w^2\,\partial_w Y + wY\,,\\
&L_1\, W = \partial_w W\,, && L_0\, W = w\,\partial_w W + \frac{1}{2}W\,, && L_{-1}\, W = w^2\,\partial_w W + wW\,.
\end{aligned} \tag{C.9}$$

They all therefore transforms as a weight 1/2 primary field of this group. This is clearly the same behavior as for any of the scalar fields, since there is no explicit w dependence in the definitions of Z, Y and W

The action of the anti-holomorphic generators is more complicated

$$\begin{aligned}
&\bar{L}_1\, Z = \partial_{\bar{w}} Z - 2Y\,, && \bar{L}_0\, Z = \bar{w}\,\partial_{\bar{w}} Z + \frac{1}{2}Z - 2\bar{w}Y\,,\\
&\bar{L}_{-1}\, Z = \bar{w}^2\,\partial_{\bar{w}} Z + \bar{w}Z - 2\bar{w}^2 Y\,,\\
&\bar{L}_1\, Y = \partial_{\bar{w}} Y - W\,, && \bar{L}_0\, Y = \bar{w}\,\partial_{\bar{w}} Y + \frac{1}{2}Y - \bar{w}W\,,\\
&\bar{L}_{-1}\, Y = \bar{w}^2\,\partial_{\bar{w}} Y + \bar{w}Y - \bar{w}^2 W\,,\\
&\bar{L}_1\, W = \partial_{\bar{w}} W\,, && \bar{L}_0\, W = \bar{w}\,\partial_{\bar{w}} W + \frac{1}{2}W\,,\\
&\bar{L}_{-1}\, W = \bar{w}^2\,\partial_{\bar{w}} Z + \bar{w}W\,.
\end{aligned} \tag{C.10}$$

likewise for $\mathrm{SU}(2)_{A'}$

$$\begin{aligned}
&R_+\, Z = 2Y\,, && R_0\, Z = 2\bar{w}Y - Z\,, && R_-\, Z = 2\bar{w}^2 Y - 2\bar{w}Z\,,\\
&R_+\, Y = W\,, && R_0\, Y = \bar{w}W\,, && R_-\, Y = \bar{w}^2 W - Z\,,\\
&R_+\, W = 0\,, && R_0\, W = W\,, && R_-\, W = 2\bar{w}W - 2Y\,,
\end{aligned} \tag{C.11}$$

and

The linear combination

$$\dot{L}_1 = \bar{L}_1 + R_+\,, \qquad \dot{L}_0 = \bar{L}_0 + R_0\,, \qquad \dot{L}_{-1} = \bar{L}_{-1} + R_-\,. \tag{C.12}$$

Has a relatively simple action on the fields

$$\begin{aligned}
&\dot{L}_1\, Z = \partial_{\bar w} Z\,, && \dot{L}_0\, Z = \bar w\,\partial_{\bar w} Z - \frac{1}{2} Z\,, && \dot{L}_{-1}\, Z = \bar w^2\,\partial_{\bar w} Z - \bar w Z\,,\\
&\dot{L}_1\, Y = \partial_{\bar w} Y\,, && \dot{L}_0\, Y = \bar w\,\partial_{\bar w} Y + \frac{1}{2} Y\,, && \dot{L}_{-1}\, Y = \bar w^2\,\partial_{\bar w} Y + \bar w Y - Z\,,\\
&\dot{L}_1\, W = \partial_{\bar w} W\,, && \dot{L}_0\, W = \bar w\,\partial_{\bar w} W + \frac{3}{2} W\,, && \dot{L}_{-1}\, W = \bar w^2\,\partial_{\bar w} W + 3\bar w W - 2Y\,.
\end{aligned} \tag{C.13}$$

Z therefore transforms as a weight $-1/2$ field of this twisted anti-holomorphic $\mathrm{SL}(2,\mathbb{R})$. Y has weight $1/2$ and W has weight $3/2$, but they are not primaries, as can be seen from the additional term in the action of $\dot{L}_{-1}$.

A different combination of generators appears as the anti-commutator of the supercharges which annihilate Z and the other supercharges. Those are

$$\hat{L}_1 = \bar{L}_1 + \frac{1}{2} R_+\,, \qquad \hat{L}_0 = \bar{L}_0 + \frac{1}{2} R_0\,, \qquad \hat{L}_{-1} = \bar{L}_{-1} + \frac{1}{2} R_-\,. \tag{C.14}$$

They act on the fields by

$$\begin{aligned}
&\hat{L}_1\, Z = \partial_{\bar w} Z - Y\,, && \hat{L}_0\, Z = \bar w\,\partial_{\bar w} Z - \bar w Y\,,\\
&\hat{L}_{-1}\, Z = \bar w^2\,\partial_{\bar w} Z - \bar w^2 Y\,, &&\\
&\hat{L}_1\, Y = \partial_{\bar w} Y - \frac{1}{2} W\,, && \hat{L}_0\, Y = \bar w\,\partial_{\bar w} Y + \frac{1}{2} Y - \frac{1}{2}\bar w W\,,\\
&\hat{L}_{-1}\, Y = \bar w^2\,\partial_{\bar w} Y + \bar w Y - \frac{1}{2}\bar w^2 W - \frac{1}{2} Z\,, &&\\
&\hat{L}_1\, W = \partial_{\bar w} W\,, && \hat{L}_0\, W = \bar w\,\partial_{\bar w} W + W\,,\\
&\hat{L}_{-1}\, W = \bar w^2\,\partial_{\bar w} W + 2\bar w W - Y\,.
\end{aligned} \tag{C.15}$$

These generators indeed arise as the anti-commutators

$$\begin{aligned}
\big\{\mathcal{Q}^+_a\,, iQ_2{}^{\dot 1 a} + \bar S^{\dot 1 \dot 2 a}\big\} &= 2\big(J^2{}_2 + \bar J^{\dot 1}{}_{\dot 1} + D\big) + \dot T^{\dot 1}{}_{\dot 1} - \dot T^{\dot 2}{}_{\dot 2} = 2(D + iJ_{12} + iR_{12}) = 4\hat{L}_0\,,\\
\big\{\mathcal{Q}^+_a\,, -iQ_2{}^{\dot 2 a}\big\} &= -2iP_{2\dot 1} - \dot T^{\dot 2}{}_{\dot 1} = P_1 + iP_2 - i(R_{23} + iR_{31}) = 2\hat{L}_1\,,\\
\big\{\mathcal{Q}^+_a\,, -\bar S^{\dot 1 \dot 1 a}\big\} &= 2iK^{\dot 1 2} + \dot T^{\dot 1}{}_{\dot 2} = K_1 - iK_2 + i(R_{23} - iR_{31}) = 2\hat{L}_{-1}\,.
\end{aligned} \tag{C.16}$$

These expressions allow one to derive relations among correlation functions of operators made out of Z, Y and W as discussed at the end of section 3.3.

D Local operators on S^2

As was mentioned in section 3, there is also a natural definition for a scalar field coupling to three scalars on S^2. At the point $x^i \in S^2$ consider the following combination of the three real scalar fields Φ^1, Φ^2 and Φ^3 (3.5)

$$Z^i = (\delta^{ij} - x^i x^j)\Phi^j + i\varepsilon_{ijk} x^j \Phi^k\,, \tag{D.1}$$

By virtue of the superscript, Z^i is a three-dimensional vector. But due to the identities

$$x^i\, Z^i = 0\,, \qquad \varepsilon_{ijk}\, x^j\, Z^k = -iZ^i\,, \tag{D.2}$$

the three different components are related by a phase.

To deal with the ambiguity it is convenient to use complex coordinates on S^2, through the stereographic projection

$$x^i = \frac{1}{1+w\bar{w}}\,(w+\bar{w},\, -i(w-\bar{w}),\, 1-w\bar{w})\,. \tag{D.3}$$

With this

$$Z^i = a^i\,\bar{a}^j\,\Phi^j\,, \qquad a^i = \frac{1}{1+w\bar{w}}\left(-i(1-w^2),\, 1+w^2,\, 2iw\right)\,. \tag{D.4}$$

So the index i on Z is related to the holomorphic coordinate w, and we can eliminate it by defining

$$Z = \frac{1}{1+w\bar{w}}\left(i(1-\bar{w}^2)\Phi^1 + (1+\bar{w}^2)\Phi^2 - 2i\bar{w}\Phi^3\right)\,. \tag{D.5}$$

This is exactly the same as (3.2), apart for a factor of $(1+w\bar{w})$ due to the conformal transformation of the fields of dimension one.

Y and W (3.32) can also be defined as

$$\begin{aligned} 2Y &= -i(x^1-ix^2)(\Phi^1+i\Phi^2) - (1+x^3)\Phi^3\,, \\ 2W &= (1+x^3)(\Phi^2 - i\Phi^1)\,. \end{aligned} \tag{D.6}$$

D.1 Supersymmetry

By use of the stereographic projection, the operators made of these fields are analogous to those on the plane and any number of operators made of Z (D.1) on the sphere will therefore share four supercharges.

For completeness we perform the supersymmetry analysis also in this case. The supersymmetry variation of Z gives

$$\delta Z \propto \bar{a}^i\rho^i\left(\epsilon_0 + x^j\gamma^j\epsilon_1\right)\,. \tag{D.7}$$

Expressing $\bar{a}^i$ and x^j in terms of w and $\bar{w}$ (or alternatively working directly with the expression (D.1)) one finds that the variation vanishes for arbitrary positions if

$$\rho^{12}\epsilon_0 + i\gamma^3\epsilon_1 = 0\,, \qquad \rho^{23}\epsilon_0 + i\gamma^1\epsilon_1 = 0\,, \qquad \rho^{31}\epsilon_0 + i\gamma^2\epsilon_1 = 0\,. \tag{D.8}$$

Eliminating ϵ_0, we find the equations

$$\gamma^{12}\epsilon_1 + \rho^{12}\epsilon_1 = 0\,, \qquad \gamma^{23}\epsilon_1 + \rho^{23}\epsilon_1 = 0\,, \qquad \gamma^{31}\epsilon_1 + \rho^{31}\epsilon_1 = 0\,. \tag{D.9}$$

This is the same as the condition for Wilson loops on S^2, equation (2.23) in [53], up to an overall sign. This means that there are solutions to these equations just like for the Wilson loops, but the combined system of loops and local operators is not supersymmetric.

To see exactly which supercharges annihilate our operators, consider again the breaking of $SU(4) \to SU(2)_{A'} \times SU(2)_{B'}$ detailed in appendix C. The combinations ρ^{ij} act as Pauli matrices of $SU(2)_{A'}$. Likewise γ^{ij} act as Pauli matrices on the chiral and anti-chiral components of the spinors. For both chiralities of ϵ_1, which we label $\epsilon_1^\pm$ equation (D.9) reads

$$(\tau^i_{L/R} + \tau^i_A)\epsilon_1^\pm = 0\,, \tag{D.10}$$

which means that $\epsilon_1^{\pm}$ is a singlet under the diagonal group $\mathrm{SU}(2)_{L/R} + \mathrm{SU}(2)_{A'}$. Explicitly, using indices $\dot{a}$ for $\mathrm{SU}(2)_{A'}$ and a for $\mathrm{SU}(2)_{B'}$ the solutions are given by the two independent two-component spinors $\epsilon^{+\,a}$ and ϵ^{-}_{a} as

$$\epsilon_{1\,\alpha}^{+\,\dot{a}a} = (\delta^2_\alpha \delta^{\dot{a}}_{\dot{1}} - \delta^1_\alpha \delta^{\dot{a}}_{\dot{2}})\epsilon^{+\,a} = i(\tau_2)^{\dot{a}}{}_{\alpha}\epsilon^{+\,a}\,, \qquad \epsilon^{-}_{1\,\dot{\alpha}\dot{a}a} = (\delta^{\dot{2}}_{\dot{\alpha}}\delta^{\dot{1}}_{\dot{a}} - \delta^{\dot{1}}_{\dot{\alpha}}\delta^{\dot{2}}_{\dot{a}})\epsilon^{-}_{a} = \varepsilon_{\dot{\alpha}\dot{a}}\epsilon^{-}_{a}\,. \tag{D.11}$$

One then solves for ϵ_0 using (D.8). Note that since these expressions have only a single γ^i matrix, they relate the ϵ_0 and ϵ_1 of opposite chiralities

$$\epsilon_0^{-\,\dot{\alpha}\dot{a}a} = i(\delta^{\dot{\alpha}}_{\dot{1}}\delta^{\dot{a}}_{\dot{2}} - \delta^{\dot{\alpha}}_{\dot{2}}\delta^{\dot{a}}_{\dot{1}})\epsilon^{+\,a} = i\varepsilon^{\dot{\alpha}\dot{a}}\epsilon^{+\,a}\,, \qquad \epsilon^{+\,\alpha}_{0\,\dot{a}a} = i(\delta^\alpha_1\delta^{\dot{2}}_{\dot{a}} - \delta^\alpha_2\delta^{\dot{1}}_{\dot{a}})\epsilon^{-}_{a} = -(\tau_2)^{\alpha}{}_{\dot{a}}\epsilon^{-}_{a}\,. \tag{D.12}$$

Using all this (and remembering the signs in (3.9)) we can write the four supersymmetry generators as

$$\mathcal{Q}_a = \bar{Q}_{\dot{1}\dot{2}a} - \bar{Q}_{\dot{2}\dot{1}a} - iS^2{}_{\dot{1}a} + iS^1{}_{\dot{2}a}\,, \qquad \dot{\mathcal{Q}}^a = Q_1{}^{\dot{2}a} - Q_2{}^{\dot{1}a} + i\bar{S}^{\dot{2}\dot{1}a} - i\bar{S}^{\dot{1}\dot{2}a}\,. \tag{D.13}$$

The anti-commutator of the two gives

$$\{\mathcal{Q}_a\,, \dot{\mathcal{Q}}^b\} = \delta^b_a(P_{1\dot{1}} + P_{2\dot{2}} + K^{\dot{1}1} + K^{\dot{2}2}) + 2iT^b{}_a = \delta^b_a(P_4 + K_4) + 2iT^b{}_a\,. \tag{D.14}$$

The trace is then the combination $P_4 + K_4$ which maps the sphere at $x^4 = 0$ to itself and the second term is the $\mathrm{SU}(2)_{B'}$ rotations, both are symmetries of all our operators Z on S^2.

We can of course also consider all the other symmetry generators and their action on these fields. Again there are certain combinations of $\mathrm{SL}(2,\mathbb{R})$ and SU(2) generators with simple actions on these fields. These are completely analogous to what is detailed in section 3.3 and appendix C.1 and we do not repeat it.

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